American-Type Exotic Options and Risk Management in Lévy-Driven Markets

Dissertation
submitted to the
Faculty of Business, Economics and Informatics
of the University of Zurich

to obtain the degree of
Doktor der Wirtschaftswissenschaften, Dr. oec.
(corresponds to Doctor of Philosophy, Ph.D.)

presented by
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approved in July 2020 at the request of
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The Faculty of Business, Economics and Informatics of the University of Zurich hereby authorizes the printing of this dissertation, without indicating an opinion of the views expressed in the work.

Zurich, 15.07.2020

Chairman of the Doctoral Board: Prof. Dr. Steven Ongena

To my beloved family

Acknowledgements

I am indebted to many people for guiding me through my Ph.D. studies. First and foremost, I would like to express my deepest gratitude to my thesis advisor, Prof. Dr. Erich Walter Farkas, for his guidance, availability and constant support in all matters. This thesis as well as the theoretical and industry experience I gained during my time at UZH would have not been possible without his help and generosity. I would also like to thank Prof. Dr. Giovanni Barone-Adesi for his advice, the helpful discussions we had on earlier versions of my work, as well as for accepting to be part of my dissertation committee. I am grateful for the time he took for me out of his busy schedule.

My gratitude is extended to my co-author Dr. Nikola Vasiljević for his passion and positivism towards research. I have truly enjoyed our collaboration as well as the various discussions we had in the last year. Furthermore, I would like to thank Dr. Robert Huitema for the opportunity to join his team at UBS in the second part of my doctoral studies. My time in his team has been very enriching, both on a professional and personal level.

During my entire journey as Ph.D. student, I was fortunate to be surrounded by smart colleagues and good friends, who greatly influenced my last three years. In particular, I would like to thank my colleagues at the university – Alex, Urban, Hanlin, Jan, Stefan, Ming, Michal, Chantal, Daria, and Lilian – as well as my colleagues at UBS – Robert, Szymon, Eduards, Andries, and Pankaj.

Last and most important, I am indebted to my family for their unconditional support and love: My parents who have spared no effort to allow me to pursue my goals in life and Zhenja who always stood by my side all these years.

Zurich, March 2020 *L.M.*

Contents

Part I

Introduction

Chapter 1

Introduction and Summary of Research Results

The present dissertation entitled "American-Type Exotic Options and Risk Management in Lévy-Driven Markets" deals with the risks and valuation of certain financial positions. These topics are at the core of mathematical finance and essential for any trading and risk management activity within the financial industry. Consequently, many of the research questions therein originate from problems faced by financial institutions in their daily business. This dissertation consists of four research articles that combine various option pricing techniques with strong Markovian arguments to study several aspects of American-type (exotic) contracts as well as related risk management problems within the important class of Lévy models. A brief summary of the four research articles (cf. [Ma19], [Ma20], [FMV19], [FM20]) is presented below.

The first research article is entitled "On Extensions of the Barone-Adesi & Whaley Method to Price American-Type Options" and deals with the numerical pricing of American standard options in certain jump-diffusion models as well as American barrier-type options under the Black & Scholes framework (cf. [BS73]). This article makes use of perturbative arguments to present a generalization of the quadratic approximation scheme proposed in [BW87] as well as of several of its extensions (cf. [Ba91], [AIL03], [CKKK07], [FMRZ15]). Here, the ansatz introduced in [FMRZ15] is extended to Merton's jump-diffusion model (cf. [Me76]) as well as to American-type barrier options under the Black & Scholes framework (cf. [BS73]). The numerical performance of the algorithm is subsequently investigated and it is shown that the resulting approximations allow for an impressive increase in accuracy when compared to the existing methods. In particular, this ansatz offers great performances for a very large range of parameters, including maturities of up to 10 years as well as in-the-money options, for which the related methods are known to fail (cf. [BW87], [Ba91], [AIL03], [CKKK07]).

The second article is entitled "Valuing Tradeability in Exponential Lévy Models" and proposes a novel theoretical way to quantify the impact of non-tradeability on the price of assets of (ordinary) exponential Lévy type. Despite the importance of tradeability for the pricing of assets, only little theoretical work has dealt with non-tradeability issues so far. Additionally, research articles studying these issues mostly derived tradeability premiums based on optimal selling strategies and could, therefore, only offer limited explanations for the existence and, in particular, the size of these premiums. This includes the works of Longstaff (cf. [Lo95], [Lo18]), as well as the articles of Koziol and Sauerbier (cf. [KS07]) and of Chesney and Kempf (cf. [CK12]). The second research article complements this literature and its current approaches by proposing a novel theoretical way to value tradeability. Here, the framework starts from an adaption of the continuous-time optional asset replacement problem initiated in the seminal paper of McDonald and Siegel (cf. [MS86]) and studies the optimal behavior of an investor that holds an asset of (ordinary) exponential

Lévy type and that faces the decision to replace it with an alternative investment project while varying the initial asset's tradeability. Based on these optimal trading strategies absolute tradeability premiums are derived and free-boundary characterizations of the latter premiums are presented. The approach is illustrated via numerical examples where various properties of the absolute tradeability premiums are discussed.

The third article, entitled "Intra-Horizon Expected Shortfall and Risk Structure in Models with Jumps", introduces a novel measure of market risk and extensively discusses its practical use. For now more than 20 years, the (point-in-time) value at risk, defined as quantile of a profit and loss distribution at the end of a predefined trading horizon, has been one of the most widely used measure of market risk for regulatory capital allocation (cf. [BCBS06], [BCBS19]). Despite its popularity, this risk measure has several major drawbacks that are all known to academics since many years. To address some of its problems, two streams have emerged in the academic literature. While certain authors (cf. [ADEH99], [RT02], [AT02]) introduced the (point-in-time) expected shortfall, a coherent risk measure that additionally depends on the tail of the underlying profit and loss distribution at the end of a predefined trading horizon, other authors (cf. [BRSW04], [Ro08], [BMK09], [BP10], [LV20]) developed a new path-dependent market risk measure, the intra-horizon value at risk. The third paper reunifies these two branches of the academic literature and combines their advantages to propose a novel, coherent and path-dependent market risk measure that depends on all extremes in a trading horizon – the intra-horizon expected shortfall. The article provides a full treatment of this risk measure by discussing various theoretical and implementation aspects. In particular, a link to simple (and maturity-randomized) first-passage probabilities of the underlying profit and loss process is provided and this is additionally used to infer diffusion and jump risk contributions to the intra-horizon expected shortfall. On the practical side, the paper proposes a simple and efficient ansatz to compute the intra-horizon risk inherent to popular Lévy dynamics. The general approach consists in combining hyper-exponential jump-diffusion approximations to (pure jump) Lévy processes having completely monotone jumps with the availability of (semi-)analytical results in this class of processes to recover approximate, though arbitrarily close intra-horizon risk results for the original process (cf. [AMP07], [Ca09], [JP10], [AR16], [LV20]). The article concludes with an empirical analysis where S&P 500 index data are calibrated to popular Lévy dynamics and the intra-horizon risk inherent to a long position in the S&P 500 index from January 1995 to April 2019 is investigated.

"Geometric Step Options with Jumps: Parity Relations, PIDEs, and Semi-Analytical Pricing" is the title of the fourth research article. This article studies geometric step options in exponential Lévy markets and contributes to the step option pricing literature in several ways. First, symmetry and parity relations for geometric double barrier step contracts under exponential Lévy models are established by generalizing the results obtained for standard options in [FM06], [FM14]. Second, various characterizations for European-type and American-type geometric double barrier step contracts as well as for their respective maturity-randomized quantities are derived. In particular, a jump-diffusion disentanglement for the early exercise premium of American-type geometric double barrier step options and its maturity-randomized equivalent is provided and diffusion and jump contributions to these early exercise premiums are characterized separately by means of partial integro-differential equations (PIDEs) and ordinary integro-differential equations (OIDEs). These results translate the formalism introduced in the third research article (cf. [FMV19]) to the setting of geometric double barrier step contracts and generalize at the same time the ideas introduced in [CYY13], [LV17] to Lévy-driven markets. Third, as an application of these characterizations, semi-analytical pricing results for (regular) European-type and American-type geometric down-and-out step call options under hyper-exponential jump-diffusion processes are derived. Finally, the early exercise structure of geometric step options once jumps are added is discussed and an analysis of the impact of jumps on the price and hedging parameters of (European-type and American-type) geometric step contracts is subsequently provided. As of now, no clear investigation of this sensitivity to jumps has been provided in the geometric step option pricing literature, which is mainly due to the scarcity of publications dealing with (American-type) geometric step options with jumps.

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Part II

Research Articles

Chapter 2

On Extensions of the Barone-Adesi & Whaley Method to Price American-Type Options

A version of this paper has been submitted to the *Journal of Computational Finance (Online ISSN: 1755-2850).*

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Abstract

The present article provides an efficient and accurate hybrid method to price American standard options in certain jump-diffusion models as well as American barrier-type options under the Black & Scholes framework. Our method generalizes the quadratic approximation scheme of Barone-Adesi & Whaley (cf. [BW87]) and several of its extensions. Using perturbative arguments, we decompose the early exercise pricing problem into sub-problems of different orders and solve these sub-problems successively. The obtained solutions are combined to recover approximations to the original pricing problem of multiple orders, with the 0-th order version matching the general Barone-Adesi & Whaley ansatz. We test the accuracy and efficiency of the approximations via numerical simulations. The results show a clear dominance of higher order approximations over their respective 0-th order version and reveal that significantly more pricing accuracy can be obtained by relying on approximations of the first few orders. Additionally, they suggest that increasing the order of any approximation by one generally refines the pricing precision, however that this happens at the expense of greater computational costs.

Keywords: American-Type Options, Exotic Options, Jump-Diffusion Models, Barone-Adesi & Whaley Approximation, Perturbation Expansion.

MSC (2010) Classification: 91-08, 91B25, 91B70, 91G20, 91G60, 91G80.

JEL Classification: C32, C63, G12, G13.

2.1 Introduction

For now more than fifty years, academics have been working on the problem of pricing American-type options. Compared to their European counterparts, these options have an early exercise feature that substantially complicates their structure and their pricing problem. Indeed, valuing American-type options is directly linked to certain types of free-boundary problems. Solving these problems is not an easy task so that analytical results are only known in few special cases. For this reason, pricing American-type options is still done in most cases via numerical approximations. Among many of the proposed approximations, hybrid approximations build a class of popular methods. These methods are based on combinations of analytical and numerical techniques and often lead to very efficient results. Examples in the case of standard American options include the class of integral representations initiated in the paper of Kim (cf. [Ki90] and [CJM92] for two prominent examples of this class) as well as the approximations proposed by MacMillan (cf. [Mc86]) and by Barone-Adesi & Whaley (cf. [BW87]) together with their extensions (cf. [Ba91], [JZ99], [KW04], [GHS09], [CS14]). An interesting survey of the main methods used for pricing standard American options (and that were developed prior to 2005) can be found in [Ba05].

More recently, continuous growth in the trading of exotic options has incentivized the development of new pricing methods for American (single) barrier options and corresponding hybrid methods have been proposed: While Guo et al. (cf. [GHS00]) and AitSahlia et al. (cf. [AIL03]) developed an approximation based on the integral representation method offered in [Ki90] and [CJM92], AitSahlia et al. (cf. [AIL03]) and Chang et al. (cf. [CKKK07]) extended the quadratic approximation of Barone-Adesi & Whaley to price American (single) barrier options. In addition to these developments in the pricing of exotics, Fatone et al. (cf. [FMRZ15]) lately proposed a novel hybrid method to price standard American options under the Black & Scholes framework (cf. [BS73]). Using perturbative arguments these authors provided a decomposition of the early exercise pricing problem into sub-problems of different orders that generalizes the Barone-Adesi & Whaley ansatz (cf. [BW87]). The present paper combines these two paths of development to offer an accurate pricing method for certain American-type options.

Our paper extends the current literature on pricing American-type options in two directions: First, we consider the problem of pricing standard American options in jump-diffusion models. Here, the ansatz introduced in [FMRZ15] is extended to a model of constant jumps as well as to Merton's jump-diffusion model (cf. [Me76]). Compared to the Black & Scholes model investigated in [FMRZ15], adding jumps to the dynamics of the asset substantially complicates the pricing attempt. In this case, early exercise of the American option may be additionally triggered by jumps and applying the Barone-Adesi & Whaley ansatz only leads to an ordinary integro-differential equation whose solution is not known in general. We solve the problem approximately by relying on similar ideas to the ones introduced in [Ba91] and provide this way a generalization of Bates' method (cf. [Ba91]). When compared to the latter method, the resulting approximations allow for a substantial increase in accuracy. In particular, our ansatz offers great performances for a very large range of parameters, including long times to maturity,[1] as well as for in-the-money options, for which Bates' method is known to fail. Secondly, we consider the problem of pricing American (single) barrier options in the model of Black & Scholes (cf. [BS73]). Here, the techniques developed in the context of standard American options are applied to extend the methods proposed in [AIL03] and [CKKK07]. On the theoretical side, our extension is substantially more challenging than these methods. This is due to the form of our expansion that, in particular, increases the complexity of the resulting equations. Here again, we provide (semi-)analytical solutions to these equations and recover approximations to the original pricing

[1] We provide numerical results for times to maturity of up to 10 years. The results are are in line with the findings obtained in [FMRZ15].

problem of multiple orders. These approximations are still very efficient in practical applications and exhibit a similar performance to the one obtained for standard American options: Compared to the simple (and modified) quadratic versions of [AIL03] and [CKKK07], our ansatz allows for a considerable increase in accuracy and the difference in accuracy between both methods is accentuated, when in-the-money options are considered. As for standard American options, this is due to the fact that the Barone-Adesi & Whaley scheme looses in accuracy when pricing in-the-money options, while this does not affect the pricing quality of our higher order versions.

Finally, we note that the same techniques can be used to extend the method proposed by [CKKK07] to price floating strike lookback options. However, since the main idea does not substantially differ from the one presented in this paper, we refrain from detailing it here. Additionally, we believe that the general idea underlying our ansatz can be combined with the results obtained in [KW04] and [CS14] to derive higher order approximations for the pricing of American-type options within the class of hyper-exponential jump-diffusions. This could be part of future work.

The remaining of this paper is structured as follows: In Section 2.2, we introduce the general framework as well as the notation used in the rest of the paper. Section 2.3 deals with the pricing problem for standard American options in jump-diffusion models: While our ansatz is first presented under general jump-diffusion assumptions, solutions to the sub-problems are subsequently derived under a model of constant jumps as well as under Merton's jump-diffusion model. The techniques developed here are then extended in Section 2.4 to deal with American (single) barrier options. All methods are finally tested in Section 2.5 and the paper concludes with Section 2.6. Complementary results are presented in the Appendices (Appendix A, B and C; Section 2.7).

2.2 General Setting and Notation

We start by introducing the general framework as well as the notation used in the rest of the paper. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$, whose filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and let $(W_t)_{t\geq 0}$ denote an $\mathbf{F}$-Brownian motion. Additionally, we let $(N_t)_{t\geq 0}$ be an $\mathbf{F}$-Poisson process with constant intensity $\lambda > 0$ and consider a financial market consisting of two assets, a deterministic savings account $(B_t(r))_{t\geq 0}$, with

$$B_t(r) = e^{rt}, \qquad r \geq 0, \, t \geq 0, \tag{2.2.1}$$

and a risky stock $(S_t)_{t\geq 0}$, whose dynamics, under a (chosen) pricing measure $\mathbb{Q}$, are described by the following stochastic differential equation (SDE)

$$dS_t = S_{t-}\left(\big(r - \delta - \lambda\zeta\big)dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}(e^{J_i} - 1)\right)\right), \qquad S_0 > 0, \tag{2.2.2}$$

with $\zeta := \mathbb{E}^{\mathbb{Q}}\left[e^{J_1} - 1\right]$. Here, the constant parameters $\delta \in \mathbb{R}$ and $\sigma > 0$ denote the dividend yield and the volatility level respectively and $\mathbb{E}^{\mathbb{Q}}[\cdot]$ refers to expectation with respect to the pricing measure $\mathbb{Q}$. Furthermore, we assume that the jump sizes $(J_i)_{i\in\mathbb{N}}$ form a sequence of independent and identically distributed random variables that are also independent of $(N_t)_{t\geq 0}$ and will denote by $f_{J_1}(\cdot)$ the density associated with the distribution of J_1. Numerous models in the financial literature belong to this framework. Important examples include the standard model of Black & Scholes (cf. [BS73]), Merton's jump-diffusion model (cf. [Me76]) as well as Kou's double exponential jump-diffusion model (cf. [Ko02]).

It is well-known that Equation (2.2.2) has a unique solution of the form

$$S_t = S_0 e^{X_t}, \qquad X_t := \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma^2\Big)t + \sigma W_t + \sum_{i=1}^{N_t} J_i, \qquad t \geq 0. \tag{2.2.3}$$

Hence, Model (2.2.2) is of (ordinary) exponential Lévy type with drift $b_X := r-\delta-\lambda\zeta-\frac{1}{2}\sigma^2+\int_{\{|y|\leq 1\}} y\,\Pi_X(dy)$, volatility $\sigma_X^2 := \sigma^2$ and jump measure given by $\Pi_X(dy) := \lambda f_{J_1}(y)dy$. We define the Lévy exponent of $(X_t)_{t\geq 0}$, $\Psi_X(\cdot)$, in the usual way and obtain that it is given, for any $\theta \in \mathbb{R}$, by

$$\begin{aligned}\Psi_X(\theta) := -\log\Big(\mathbb{E}^{\mathbb{Q}}\Big[e^{i\theta X_1}\Big]\Big) &= -ib_X\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} \big(1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\leq 1\}}\big)\Pi_X(dy) \\ &= -i\Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma^2\Big)\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\int_{\mathbb{R}} \big(1 - e^{i\theta y}\big) f_{J_1}(y)dy. \end{aligned} \tag{2.2.4}$$

Similarly, its Laplace exponent, $\Phi_X(\cdot)$, is well-defined for any $\theta \in \mathbb{R}$ satisfying $\mathbb{E}^{\mathbb{Q}}\big[e^{\theta J_1}\big] < \infty$ and is recovered from $\Psi_X(\cdot)$ via the following relation:

$$\Phi_X(\theta) := -\Psi_X(-i\theta) = \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma^2\Big)\theta + \frac{1}{2}\sigma^2\theta^2 - \lambda\int_{\mathbb{R}} \big(1 - e^{\theta y}\big) f_{J_1}(y)dy. \tag{2.2.5}$$

Finally, it should be noticed that $(S_t)_{t\geq 0}$ has a Markovian structure. Following standard theory for Markov processes, we therefore obtain that its infinitesimal generator is a partial integro-differential operator given, for sufficiently smooth $V : [0,\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, by

$$\begin{aligned}\mathcal{A}_S V(\mathcal{T},x) &:= \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathbb{Q}}\big[V(\mathcal{T},S_t)\big] - V(\mathcal{T},x)}{t} \\ &= \frac{1}{2}\sigma^2 x^2 \partial_x^2 V(\mathcal{T},x) + \big(r - \delta - \lambda\zeta\big)x\partial_x V(\mathcal{T},x) + \lambda\int_{\mathbb{R}} \big(V(\mathcal{T},xe^y) - V(\mathcal{T},x)\big) f_{J_1}(y)dy, \end{aligned} \tag{2.2.6}$$

where $\mathbb{E}_x^{\mathbb{Q}}[\cdot]$ denotes expectation under $\mathbb{Q}_x$, the pricing measure having initial distribution $S_0 = x$. We will extensively make use of these notations in the upcoming sections.

2.3 Approximation of Standard American Options

We first consider the problem of pricing standard American options and derive an approximation that generalizes the ansatz adopted by Barone-Adesi & Whaley in the standard Black & Scholes model (cf. [BW87]) and extended by Bates to Merton's jump-diffusion model (cf. [Ba91]). Our derivations focus on the standard American call. However, we note that the case of a standard American put can be treated analogously and only requires few obvious adjustements. For this reason, our numerical discussion in Section 2.5 also provides simulation results for American put options.

2.3.1 Pricing Problem and Perturbation Expansion

We start by reviewing few well-known facts on pricing standard American (call) options in models of the type of (2.2.1), (2.2.2). First, we recall that the value of a standard American call option on $(S_t)_{t\geq 0}$ having maturity $\mathcal{T} \geq 0$, initial value $S_0 = x \geq 0$ and strike price $K \geq 0$ has the following representation

$$\mathcal{C}_A(\mathcal{T},x;K) := \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}_x^{\mathbb{Q}}\left[B_\tau(r)^{-1}\left(S_\tau - K\right)^+\right], \tag{2.3.1}$$

where $\mathfrak{T}_{[0,\mathcal{T}]}$ denotes the set of stopping times that take values in the interval $[0,\mathcal{T}]$, and that its European counterpart is obtained via

$$\mathcal{C}_E(\mathcal{T},x;K) := \mathbb{E}_x^{\mathbb{Q}}\left[B_{\mathcal{T}}(r)^{-1}\left(S_{\mathcal{T}}-K\right)^+\right]. \tag{2.3.2}$$

Although European Option (2.3.2) has, for many densities $f_{J_1}(\cdot)$, a closed form representation, pricing American-style derivatives is not an easy task and is usually done via numerical approximations. One popular way to derive such approximations consists in decomposing Option (2.3.1) into two components: Its European counterpart (2.3.2) and an early exercise premium $\mathcal{E}(\cdot)$, obtained via

$$\mathcal{E}(\mathcal{T},x;K) := \mathcal{C}_A(\mathcal{T},x;K) - \mathcal{C}_E(\mathcal{T},x;K). \tag{2.3.3}$$

This decomposition is of great practical interest, since it usually reduces the pricing problem to the valuation of the early exercise premium $\mathcal{E}(\cdot)$ and, therefore, leads for a particular approximation method to a higher pricing accuracy when compared with a direct application of the same method to (2.3.1) instead. We will also adopt this approach and derive an approximation of the early exercise premium $\mathcal{E}(\cdot)$ for finite maturities, i.e. we fix a final maturity $T>0$ and will focus on the valuation of $\mathcal{E}(\mathcal{T},x;K)$ for $\mathcal{T}\in[0,T]$.[2] To this end, we note by the same arguments as the ones provided in [Ma20] that the early exercise premium $\mathcal{E}(\cdot)$ is linked to a partial integro-differential equation (PIDE) and has the following properties:

1. If $\delta \le 0$, the early exercise premium $\mathcal{E}(\cdot)$ satisfies

$$\mathcal{E}(\mathcal{T},x;K)=0, \qquad \forall(\mathcal{T},x)\in[0,T]\times[0,\infty).$$

2. If $\delta > 0$, the pair $\big(\mathcal{E}(\cdot),\mathfrak{b}(\cdot)\big)$, where $\mathfrak{b}(\cdot)$ denotes the (corresponding) early exercise boundary, is a solution of the following free-boundary problem:

$$-\partial_{\mathcal{T}}\mathcal{E}(\mathcal{T},x;K) + \mathcal{A}_S\mathcal{E}(\mathcal{T},x;K) - r\mathcal{E}(\mathcal{T},x;K) = 0, \quad x\in(0,\mathfrak{b}(\mathcal{T})),\ \mathcal{T}\in(0,T], \tag{2.3.4}$$

subject to the boundary conditions

$$\mathcal{E}(\mathcal{T},\mathfrak{b}(\mathcal{T});K) = \mathfrak{b}(\mathcal{T}) - K - \mathcal{C}_E(\mathcal{T},\mathfrak{b}(\mathcal{T});K), \qquad \mathcal{T}\in(0,T], \tag{2.3.5}$$

$$\partial_x\mathcal{E}(\mathcal{T},\mathfrak{b}(\mathcal{T});K) = 1 - \partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}(\mathcal{T});K), \qquad \mathcal{T}\in(0,T], \tag{2.3.6}$$

$$\mathcal{E}(\mathcal{T},0;K) = 0, \qquad \mathcal{T}\in(0,T], \tag{2.3.7}$$

and initial condition

$$\mathcal{E}(0,x;K) = 0, \qquad x\in(0,\mathfrak{b}(\mathcal{T})). \tag{2.3.8}$$

Therefore, we focus from now on on the non-trivial case 2. and derive a solution to the above free-boundary characterization.

2.3.1.1 The Barone-Adesi & Whaley Ansatz

As done in [BW87], we next rewrite the early exercise premium $\mathcal{E}(\cdot)$ in the following form

$$\mathcal{E}(\mathcal{T},x;K) = h(\mathcal{T})F(h(\mathcal{T}),x;K), \tag{2.3.9}$$

where $h(\mathcal{T}) := 1-e^{-r\mathcal{T}}$ and $F(\cdot)$ is an auxiliary "well-behaved" function that will be determined later. Under this representation, straightforward computations transform Equations (2.3.4)-(2.3.7) into a new problem:

$$-\frac{r}{h(\mathcal{T})}F(h(\mathcal{T}),x;K) + \mathcal{A}_SF(h(\mathcal{T}),x;K) - r\big(1-h(\mathcal{T})\big)\partial_hF(h(\mathcal{T}),x;K) = 0, \quad x\in(0,\mathfrak{b}(\mathcal{T})),\ \mathcal{T}\in(0,T], \tag{2.3.10}$$

[2] We understand this value as the time-t value of the early exercise premium by having in mind that $\mathcal{T} = T - t$.

with boundary conditions

$$F(h(\mathcal{T}),\mathfrak{b}(\mathcal{T});K) = \frac{1}{h(\mathcal{T})}\Big(\mathfrak{b}(\mathcal{T}) - K - \mathcal{C}_E(\mathcal{T},\mathfrak{b}(\mathcal{T});K)\Big), \quad \mathcal{T} \in (0,T], \tag{2.3.11}$$

$$\partial_x F(h(\mathcal{T}),\mathfrak{b}(\mathcal{T});K) = \frac{1}{h(\mathcal{T})}\Big(1 - \partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}(\mathcal{T});K)\Big), \quad \mathcal{T} \in (0,T], \tag{2.3.12}$$

$$F(h(\mathcal{T}),0;K) = 0, \quad \mathcal{T} \in (0,T]. \tag{2.3.13}$$

Although Initial Condition (2.3.8) is not anymore required to hold, it will be naturally satisfied whenever $F(\cdot)$ has "good properties".[3] This follows, from (2.3.9), since $h(0) = 0$ clearly holds.

Starting from a problem that corresponds to (2.3.10)-(2.3.13) in the Black & Scholes model, the authors in [BW87] had the brilliant idea to drop the last term in their partial differential equation (PDE) corresponding to (2.3.10). This allowed them to convert the initial problem into a much more manageable one[4], for which an "analytical solution"[5] can be easily derived (cf. [BW87]). Also in our case this approach can be taken. However, omitting the last term $r\big(1 - h(\mathcal{T})\big)\partial_h F(h(\mathcal{T}),x;K)$ in Equation (2.3.10) now transforms it into an ordinary integro-differential equation (OIDE) whose solution is not known in general. Nevertheless, approximate solutions have proven to be effective in some types of model. Such an approximation was first introduced under Merton's jump-diffusion model in [Ba91]. An application of the same ansatz under a model of constant jumps (cf. Equation (2.3.35)) is also presented in [JC04], [JYC06].

2.3.1.2 Generalization of the Barone-Adesi & Whaley Ansatz

Instead of relying on the well-known Barone-Adesi & Whaley ansatz, we follow an extended approach to Problem (2.3.10)-(2.3.13) that was proposed in the classical Black & Scholes model in [FMRZ15]. To this end, we introduce a new parameter, a perturbation parameter $\epsilon \in [0,1]$, in (2.3.10)-(2.3.13) and consider the following modified problem:

$$-\frac{r}{h(\mathcal{T})}F^\epsilon(h(\mathcal{T}),x;K) + \mathcal{A}_S F^\epsilon(h(\mathcal{T}),x;K) - \epsilon r\big(1-h(\mathcal{T})\big)\partial_h F^\epsilon(h(\mathcal{T}),x;K) = 0, \quad x \in (0,\mathfrak{b}^\epsilon(\mathcal{T})), \mathcal{T} \in (0,T], \tag{2.3.14}$$

with boundary conditions

$$F^\epsilon(h(\mathcal{T}),\mathfrak{b}^\epsilon(\mathcal{T});K) = \frac{1}{h(\mathcal{T})}\Big(\mathfrak{b}^\epsilon(\mathcal{T}) - K - \mathcal{C}_E(\mathcal{T},\mathfrak{b}^\epsilon(\mathcal{T});K)\Big), \quad \mathcal{T} \in (0,T], \tag{2.3.15}$$

$$\partial_x F^\epsilon(h(\mathcal{T}),\mathfrak{b}^\epsilon(\mathcal{T});K) = \frac{1}{h(\mathcal{T})}\Big(1 - \partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}^\epsilon(\mathcal{T});K)\Big), \quad \mathcal{T} \in (0,T], \tag{2.3.16}$$

$$F^\epsilon(h(\mathcal{T}),0;K) = 0, \quad \mathcal{T} \in (0,T]. \tag{2.3.17}$$

Switching from (2.3.10)-(2.3.13) to the new problem (2.3.14)-(2.3.17) clearly allows for a more general treatment of the pricing attempt. Indeed, for $\epsilon = 1$, PIDEs (2.3.14) and (2.3.10) are identical and the perturbative approach reduces to the original problem. Additionally, solving the modified problem while taking $\epsilon = 0$ allows to recover the classical Barone-Adesi & Whaley ansatz.

In order to solve Problem (2.3.14)-(2.3.17), we make use of a typical perturbative ansatz (cf. [Ve05]) and

[3] The ansatz we will follow consists in representing $F(\cdot)$ by a series of products of logarithms and power functions. Consequently, $F(\cdot)$ will have sufficiently good properties.

[4] Under the Black & Scholes model the resulting equation simplifies to an ordinary differential equation (ODE).

[5] This solution still depends on the free-boundary $\mathfrak{b}(\cdot)$. Finding this boundary level requires however the use of numerical methods.

assume that the solution pair $\big(F^\epsilon(\cdot), \mathfrak{b}^\epsilon(\cdot)\big)$ to Equations (2.3.14)-(2.3.17) has, for any $\epsilon \in [0,1]$, a representation as "well-behaved"[6] series expansion of the form

$$F^\epsilon(h(\mathcal{T}), x; K) = \sum_{n=0}^{\infty} \epsilon^n f_n(h(\mathcal{T}), x; K), \qquad x \in (0, \mathfrak{b}^\epsilon(\mathcal{T})), \mathcal{T} \in [0, T], \tag{2.3.18}$$

$$\mathfrak{b}^\epsilon(\mathcal{T}) = \sum_{n=0}^{\infty} \epsilon^n b_n(\mathcal{T}), \qquad \mathcal{T} \in [0, T], \tag{2.3.19}$$

for some functions $\big(f_n(\cdot)\big)_{n\in\mathbb{N}_0}$ and $\big(b_n(\cdot)\big)_{n\in\mathbb{N}_0}$. Additionally, we define partial sums of N-th order via

$$F_N^\epsilon(h(\mathcal{T}), x; K) = \sum_{n=0}^{N} \epsilon^n f_n(h(\mathcal{T}), x; K), \qquad x \in (0, \mathfrak{b}_N^\epsilon(\mathcal{T})), \mathcal{T} \in [0, T], \tag{2.3.20}$$

$$\mathfrak{b}_N^\epsilon(\mathcal{T}) = \sum_{n=0}^{N} \epsilon^n b_n(\mathcal{T}), \qquad \mathcal{T} \in [0, T]. \tag{2.3.21}$$

Using Representation (2.3.18), we obtain upon setting $f_{-1}(h(\mathcal{T}), x; K) := 0$ that

$$\sum_{n=0}^{\infty} \epsilon^n \left(-\frac{r}{h(\mathcal{T})} f_n(h(\mathcal{T}), x; K) + \mathcal{A}_S f_n(h(\mathcal{T}), x; K) - r\big(1 - h(\mathcal{T})\big)\partial_h f_{n-1}(h(\mathcal{T}), x; K) \right) = 0 \tag{2.3.22}$$

and imposing this equation to hold order by order in the powers of ϵ leads to the following recurrent system of n-th order problems: For $n = 0$, the 0-th order problem reads

$$-\frac{r}{h(\mathcal{T})} f_0(h(\mathcal{T}), x; K) + \mathcal{A}_S f_0(h(\mathcal{T}), x; K) = 0, \qquad x \in (0, \mathfrak{b}_0^\epsilon(\mathcal{T})), \mathcal{T} \in (0, T], \tag{2.3.23}$$

with boundary conditions

$$f_0(h(\mathcal{T}), \mathfrak{b}_0^\epsilon(\mathcal{T}); K) = \frac{1}{h(\mathcal{T})}\Big(\mathfrak{b}_0^\epsilon(\mathcal{T}) - K - \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_0^\epsilon(\mathcal{T}); K)\Big), \qquad \mathcal{T} \in (0, T], \tag{2.3.24}$$

$$\partial_x f_0(h(\mathcal{T}), \mathfrak{b}_0^\epsilon(\mathcal{T}); K) = \frac{1}{h(\mathcal{T})}\Big(1 - \partial_x \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_0^\epsilon(\mathcal{T}); K)\Big), \qquad \mathcal{T} \in (0, T], \tag{2.3.25}$$

$$f_0(h(\mathcal{T}), 0; K) = 0, \qquad \mathcal{T} \in (0, T]. \tag{2.3.26}$$

Additionally, the following higher order problems ($n \in \mathbb{N}$) are obtained:

$$-\frac{r}{h(\mathcal{T})} f_n(h(\mathcal{T}), x; K) + \mathcal{A}_S f_n(h(\mathcal{T}), x; K) - r\big(1 - h(\mathcal{T})\big)\partial_h f_{n-1}(h(\mathcal{T}), x; K) = 0, \quad x \in (0, \mathfrak{b}_n^\epsilon(\mathcal{T})), \mathcal{T} \in (0, T], \tag{2.3.27}$$

with boundary conditions, for $\mathcal{T} \in (0, T]$:

$$f_n(h(\mathcal{T}), \mathfrak{b}_n^\epsilon(\mathcal{T}); K) = \frac{\epsilon^{-n}}{h(\mathcal{T})}\Big(\mathfrak{b}_n^\epsilon(\mathcal{T}) - K - \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_n^\epsilon(\mathcal{T}); K) - h(\mathcal{T}) F_{n-1}^\epsilon(h(\mathcal{T}), \mathfrak{b}_n^\epsilon(\mathcal{T}); K)\Big), \tag{2.3.28}$$

$$\partial_x f_n(h(\mathcal{T}), \mathfrak{b}_n^\epsilon(\mathcal{T}); K) = \frac{\epsilon^{-n}}{h(\mathcal{T})}\Big(1 - \partial_x \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_n^\epsilon(\mathcal{T}); K) - h(\mathcal{T}) \partial_x F_{n-1}^\epsilon(h(\mathcal{T}), \mathfrak{b}_n^\epsilon(\mathcal{T}); K)\Big), \tag{2.3.29}$$

$$f_n(h(\mathcal{T}), 0; K) = 0. \tag{2.3.30}$$

[6] In particular, we will assume that any derivative of $F^\epsilon(\cdot)$ can be obtained by differentiating inside the sum.

Recall that our initial early exercise premium valuation attempt is related to the above problems via the following relation

$$\mathcal{E}(\mathcal{T},x;K) = h(\mathcal{T})F^{\epsilon=1}(h(\mathcal{T}),x;K) =: \mathcal{E}^{\epsilon=1}(\mathcal{T},x;K). \tag{2.3.31}$$

Assuming that $F^{\epsilon}(\cdot)$ has a representation of the form of (2.3.18), we therefore expect to obtain N-th order approximations of the early exercise premium by means of the following quantities:

$$\mathcal{E}_N^{\epsilon=1}(\mathcal{T},x;K) := h(\mathcal{T})F_N^{\epsilon=1}(h(\mathcal{T}),x;K) = h(\mathcal{T})\sum_{n=0}^{N} f_n(h(\mathcal{T}),x;K), \qquad x \in [0,\mathfrak{b}_N^{\epsilon=1}(\mathcal{T})), \tag{2.3.32}$$

$$\mathcal{E}_N^{\epsilon=1}(\mathcal{T},x;K) := x - K - \mathcal{C}_E(\mathcal{T},x;K), \qquad x \in [\mathfrak{b}_N^{\epsilon=1}(\mathcal{T}),\infty), \tag{2.3.33}$$

where $\mathcal{T} \in [0,T]$. Consequently, we focus in the sequel on the n-th order problems (2.3.23)-(2.3.26) and (2.3.27)-(2.3.30) for $\epsilon = 1$ and will subsequently recover approximations of multiple orders via (2.3.32), (2.3.33).

At this point, we should note that the boundary functions $(b_n(\cdot))_{n\in\mathbb{N}_0}$ play no role in the respective n-th order problems. Indeed, numerical experiments have shown that the partial sums of the first few orders computed by solving the n-th order problems applied directly to $(\mathfrak{b}_n^{\epsilon=1}(\cdot))_{n\in\mathbb{N}_0}$ provide better results than the corresponding partial sums obtained by solving the same problems but applied order by order to $(b_n(\cdot))_{n\in\mathbb{N}_0}$.[7] Therefore, solving the n-th order problems will always be carried out directly in terms of $(\mathfrak{b}_n^{\epsilon=1}(\cdot))_{n\in\mathbb{N}_0}$.

2.3.2 Solutions under Constant Jumps

We next turn to the derivation of N-th order approximations under constant jumps, i.e. we fix $\varphi \in \mathbb{R}$ and assume throughout the rest of this section that the jump measure Π_X is given by

$$\lambda f_{J_1}(y)dy = \Pi_X(dy) = \lambda\delta_\varphi(dy), \tag{2.3.34}$$

where $\delta_\varphi(\cdot)$ denotes the Dirac measure at φ. This is equivalent to the assumption that the asset dynamics $(S_t)_{t\geq 0}$ evolve, under the pricing measure $\mathbb{Q}$, according to the following SDE

$$dS_t = S_{t-}\Big(\big(r-\delta-\lambda(e^\varphi-1)\big)dt + \sigma dW_t + (e^\varphi-1)dN_t\Big), \qquad S_0 > 0, \tag{2.3.35}$$

where the processes $(W_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ and the parameters $\lambda > 0$, $r \geq 0$, $\delta \in \mathbb{R}$ and $\sigma > 0$ have the same properties as in (2.2.1), (2.2.2). In this case, $(S_t)_{t\geq 0}$ is recovered from (2.2.3) with

$$X_t := \Big(r-\delta-\lambda(e^\varphi-1)-\frac{1}{2}\sigma^2\Big)t + \sigma W_t + \varphi N_t, \qquad t \geq 0,$$

and its infinitesimal generator takes the following simplified form

$$\mathcal{A}_S V(\mathcal{T},x) = \frac{1}{2}\sigma^2x^2\partial_x^2 V(\mathcal{T},x) + \big(r-\delta-\lambda(e^\varphi-1)\big)x\partial_x V(\mathcal{T},x) + \lambda\big(V(\mathcal{T},xe^\varphi) - V(\mathcal{T},x)\big). \tag{2.3.36}$$

Whenever $\varphi \leq 0$, this will in particular allows us to derive a well-known solution to the OIDE arising in the 0-th order problem, as it now simplifies in the continuation region to an homogeneous second order linear ODE that does not depend anymore on boundary terms. Analogously, deriving an exact solution of the OIDE arising in the 0-th order problem for the American put requires that $\varphi \geq 0$. This will be outlined in the next section.

[7] This is in line with the findings in [FMRZ15].

2.3.2.1 Solution of the 0-th Order Problem

We start our derivations by noting that, under Model (2.3.35), the Laplace exponent of $(X_t)_{t\geq 0}$, $\Phi_X(\cdot)$, is well-defined for all $\theta \in \mathbb{R}$. Furthermore, it can be easily seen that $\theta \mapsto \Phi_X(\theta)$ is convex and satisfies $\Phi_X(0) = 0$ and $\lim\limits_{|\theta|\to\infty} \Phi_X(\theta) = \infty$. Therefore, the equation $\Phi_X(\theta) = y$ has for any $y > 0$ two solutions, a positive and a negative root. We will denote by $\Phi_X^{-1,+}(y)$ its positive root and by $\Phi_X^{-1,-}(y)$ its negative root.

We now turn to the 0-th order problem. For $\varphi \leq 0$, Equation (2.3.23) is well-known and its general solution takes the simple form

$$f_0(h(\mathcal{T}), x; K) = c_{0,0}^{+}(h(\mathcal{T}))x^{\rho_+(h(\mathcal{T}))} + c_{0,0}^{-}(h(\mathcal{T}))x^{\rho_-(h(\mathcal{T}))}, \qquad x \in (0, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T})),\ \mathcal{T} \in (0, T], \tag{2.3.37}$$

where, for $\mathcal{T} \in (0, T]$,

$$\rho_+(h(\mathcal{T})) := \Phi_X^{-1,+}\Big(\frac{r}{h(\mathcal{T})}\Big), \qquad \rho_-(h(\mathcal{T})) := \Phi_X^{-1,-}\Big(\frac{r}{h(\mathcal{T})}\Big), \tag{2.3.38}$$

and $c_{0,0}^{+}(h(\mathcal{T}))$ and $c_{0,0}^{-}(h(\mathcal{T}))$ are "constants" to be determined. Conversely, the ODE corresponding to (2.3.23) under this model takes a special form immediately below the exercise boundary when $\varphi > 0$. Indeed, for $x \in [\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})e^{-\varphi}, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T}))$, Equation (2.3.23) becomes

$$\begin{aligned} -\frac{r}{h(\mathcal{T})} f_0(h(\mathcal{T}), x; K) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 f_0(h(\mathcal{T}), x; K) + \big(r - \delta - \lambda(e^{\varphi} - 1)\big) x \partial_x f_0(h(\mathcal{T}), x; K) \\ + \lambda\Big(\frac{1}{h(\mathcal{T})}\big(xe^{\varphi} - K - \mathcal{C}_E(\mathcal{T}, xe^{\varphi}; K)\big) - f_0(h(\mathcal{T}), x; K)\Big) = 0 \end{aligned}$$

and, unfortunately, there is no known solution to this equation.[8] Since we expect $f_0(\cdot)$ to be continuous in the jump parameter φ, it appears however sensible to approximate the solution for "small" jump sizes anyway via (2.3.37). This is in line with the approximation proposed in [Ba91] and with the discussion in [JC04]. We will also follow this approach and provide numerical tests to the resulting N-th order approximations in Section 2.5.

To derive an expression for $c_{0,0}^{+}(\cdot)$, $c_{0,0}^{-}(\cdot)$ and $\mathfrak{b}_0^{\epsilon=1}(\cdot)$, we use the complementary conditions (2.3.24), (2.3.25) and (2.3.26). First, we note that (2.3.26) implies that $c_{0,0}^{-}(h(\mathcal{T})) \equiv 0$. Secondly, substituting (2.3.37) into Condition (2.3.25), allows us to express $c_{0,0}^{+}(\cdot)$ in terms of the free-boundary $\mathfrak{b}_0^{\epsilon=1}(\cdot)$ as

$$c_{0,0}^{+}(h(\mathcal{T})) = \frac{\Big(\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\Big)^{1-\rho_+(h(\mathcal{T}))}}{h(\mathcal{T})\rho_+(h(\mathcal{T}))}\Big(1 - \partial_x \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T}); K)\Big), \qquad \mathcal{T} \in (0, T]. \tag{2.3.39}$$

Finally, substituting again (2.3.37) into (2.3.24) and inserting Representation (2.3.39) in the resulting equation, gives

$$\mathfrak{b}_0^{\epsilon=1}(\mathcal{T}) = K + \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T}); K) + \frac{\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})}{\rho_+(h(\mathcal{T}))}\Big(1 - \partial_x \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T}); K)\Big), \qquad \mathcal{T} \in (0, T], \tag{2.3.40}$$

a non-linear equation in $\mathfrak{b}_0^{\epsilon=1}(\cdot)$. Therefore, solving Equation (2.3.40) numerically for $\mathcal{T} \in (0, T]$ gives $\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})$ and subsequently allows us to recover $c_{0,0}^{+}(h(\mathcal{T}))$ via Relation (2.3.39) to finally obtain the 0-th order premium $f_0(\cdot)$.

[8] When considering an American put option, $\varphi < 0$ transforms (2.3.23) for any $x \in (\mathfrak{b}_0^{\epsilon=1}(\mathcal{T}), \mathfrak{b}_0^{\epsilon=1}(\mathcal{T})e^{-\varphi}]$ into a similar equation. Here again, there is no known solution to the resulting equation.

2.3.2.2 Solution of the Higher Order Problems

We now turn to the higher order problems, i.e. we seek, for $n \in \mathbb{N}$, a solution to (2.3.27)-(2.3.30). Generalizing the form of the solution obtained in the 0-th order problem, we make the following ansatz:

$$f_n(h(\mathcal{T}), x; K) = \left(c_{n,0}^{+}(h(\mathcal{T})) + \sum_{j=1}^{2n} c_{n,j}^{+}(h(\mathcal{T})) \log(x)^j\right) x^{\rho_+(h(\mathcal{T}))}, \qquad x \in (0, \mathfrak{b}_n^{\epsilon=1}(\mathcal{T})),\ \mathcal{T} \in (0, T], \quad (2.3.41)$$

where the "constants" $c_{n,0}^{+}(\cdot)$ and, for $j \in \{1, 2, \ldots, 2n\}$, $c_{n,j}^{+}(\cdot)$ are still to determine. Whether or not this ansatz provides good results is an issue that we will consider in the numerical simulations of Section 2.5. However, it gives a convenient way to solve (2.3.27)-(2.3.30), since it allows us to obtain a system of linear equations in the coefficients $c_{n,j}^{+}(\cdot)$ that can be solved using standard numerical methods. For the derivation of this system, we substitute (2.3.41) into PDE (2.3.27), use Property (2.3.38), and match the powers of the logarithm. This gives, for $n \in \mathbb{N}$, the following system of $2n$ linear equations in the $2n$ unknowns $\left(c_{n,j}^{+}(h(\mathcal{T}))\right)_{j \in \{1,\ldots,2n\}}$:

$$\begin{aligned} 2n\left[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T})) - 1\big) + r - \delta + \lambda\big(\varphi e^{\rho_+(h(\mathcal{T}))\varphi} - (e^{\varphi} - 1)\big)\right] & c_{n,2n}^{+}(h(\mathcal{T})) \\ &= r(1 - h(\mathcal{T}))\partial_h \rho_+(h(\mathcal{T})) c_{n-1,2(n-1)}^{+}(h(\mathcal{T})), \end{aligned} \quad (2.3.42)$$

$$\begin{aligned} j\left[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T})) - 1\big) + r - \delta + \lambda\big(\varphi e^{\rho_+(h(\mathcal{T}))\varphi} - (e^{\varphi} - 1)\big)\right] & c_{n,j}^{+}(h(\mathcal{T})) \\ + j(j+1)\frac{\sigma^2}{2} c_{n,j+1}^{+}(h(\mathcal{T})) + \lambda \sum_{k=j+1}^{2n} \binom{k}{j-1} & \varphi^{k-(j-1)} e^{\rho_+(h(\mathcal{T}))\varphi} c_{n,k}^{+}(h(\mathcal{T})) \\ &= r(1 - h(\mathcal{T}))\big(\partial_h c_{n-1,j-1}^{+}(h(\mathcal{T})) + \partial_h \rho_+(h(\mathcal{T})) c_{n-1,j-2}^{+}(h(\mathcal{T}))\big), \end{aligned} \quad (2.3.43)$$

$$\begin{aligned} \left[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T})) - 1\big) + r - \delta + \lambda\big(\varphi e^{\rho_+(h(\mathcal{T}))\varphi} - (e^{\varphi} - 1)\big)\right] & c_{n,1}^{+}(h(\mathcal{T})) \\ + \sigma^2 c_{n,2}^{+}(h(\mathcal{T})) + \lambda \sum_{k=2}^{2n} \varphi^k e^{\rho_+(h(\mathcal{T}))\varphi} c_{n,k}^{+}(h(\mathcal{T})) &= r(1 - h(\mathcal{T}))\partial_h c_{n-1,0}^{+}(h(\mathcal{T})), \end{aligned} \quad (2.3.44)$$

where Equation (2.3.43) only holds for $j \in \{2, 3, \ldots, 2n-1\}$.

To conclude, we proceed as in the derivation of the 0-th order approximation and derive, for any $n \in \mathbb{N}$, an expression for both $c_{n,0}^{+}(\cdot)$ and $\mathfrak{b}_n^{\epsilon=1}(\cdot)$ by substituting (2.3.41) into Equations (2.3.28) and (2.3.29). This first allows us to express $c_{n,0}^{+}(h(\mathcal{T}))$, for $\mathcal{T} \in (0, T]$, in terms of the free boundary $\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})$ as

$$\begin{aligned} c_{n,0}^{+}(h(\mathcal{T})) = \frac{\left(\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})\right)^{1-\rho_+(h(\mathcal{T}))}}{h(\mathcal{T})\rho_+(h(\mathcal{T}))} & \left(1 - \partial_x \mathcal{C}_E(\mathcal{T}, \mathfrak{b}_n^{\epsilon=1}(\mathcal{T}); K) - h(\mathcal{T})\partial_x F_{n-1}^{\epsilon=1}(h(\mathcal{T}), \mathfrak{b}_n^{\epsilon=1}(\mathcal{T}); K)\right) \\ - \frac{1}{\rho_+(h(\mathcal{T}))} \sum_{j=1}^{2n} c_{n,j}^{+}(h(\mathcal{T})) j \log\left(\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})\right)^{j-1} & - \sum_{j=1}^{2n} c_{n,j}^{+}(h(\mathcal{T})) \log\left(\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})\right)^{j}, \end{aligned} \quad (2.3.45)$$

and to finally obtain a characterization of the free boundary $\mathfrak{b}_n^{\epsilon=1}(\cdot)$ by means of the following non-linear equation:

$$\begin{aligned}\mathfrak{b}_n^{\epsilon=1}(\mathcal{T}) &= K + \mathcal{C}_E(\mathcal{T},\mathfrak{b}_n^{\epsilon=1}(\mathcal{T});K) + h(\mathcal{T})F_{n-1}^{\epsilon=1}(h(\mathcal{T}),\mathfrak{b}_n^{\epsilon=1}(\mathcal{T});K)\\ &\quad+\frac{\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})}{\rho_+(h(\mathcal{T}))}\Big(1-\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_n^{\epsilon=1}(\mathcal{T});K) - h(\mathcal{T})\partial_x F_{n-1}^{\epsilon=1}(h(\mathcal{T}),\mathfrak{b}_n^{\epsilon=1}(\mathcal{T});K)\Big)\\ &\quad-\frac{\big(\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})\big)^{\rho_+(h(\mathcal{T}))}h(\mathcal{T})}{\rho_+(h(\mathcal{T}))}\sum_{j=1}^{2n} c_{n,j}^+(h(\mathcal{T}))j\log\big(\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})\big)^{j-1}. \qquad (2.3.46)\end{aligned}$$

Once again, this equation can be solved for any $\mathcal{T}\in(0,T]$ using standard numerical techniques to derive $\mathfrak{b}_n^{\epsilon=1}(\mathcal{T})$, $c_{n,0}^+(h(\mathcal{T}))$ and ultimately the n-th order premium $f_n(\cdot)$.

Remark 2.1.

i) As seen from Equations (2.3.42)-(2.3.44), our higher order approximations ($n\in\mathbb{N}$) depend on the derivatives $\partial_h\rho_+(\cdot)$ and $\partial_h c_{k,j}^+(\cdot)$ for $j=0,\dots,2(n-1)$ and $k=0,\dots,n-1$. Although these derivatives can be implemented via (central) finite differencing, one may want to increase the stability of certain results. This can be achieved by deriving corresponding (non-linear) equations from (2.3.38)-(2.3.40) and (2.3.42)-(2.3.46). For instance, differentiating Equation (2.3.38) gives that $\partial_h\rho_+(h(\mathcal{T}))$ solves, for any $\mathcal{T}\in(0,T]$, the following equation:

$$\big(r-\delta-\lambda(e^{\varphi}-1)-\frac{1}{2}\sigma^2\big)\partial_h\rho_+(h(\mathcal{T}))+\sigma^2\rho_+(h(\mathcal{T}))\partial_h\rho_+(h(\mathcal{T}))+\lambda\varphi\partial_h\rho_+(h(\mathcal{T}))e^{\varphi\rho_+(h(\mathcal{T}))}=-\frac{r}{h(\mathcal{T})^2}.$$

Similarly, one obtains from Equations (2.3.39) and (2.3.40) that, for any $\mathcal{T}\in(0,T]$,

$$\begin{aligned}\partial_h c_{0,0}^+(h(\mathcal{T})) &= \frac{\Big(\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\Big)^{1-\rho_+(h(\mathcal{T}))}}{re^{-r\mathcal{T}}}\Bigg[\frac{\frac{\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})}{\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})}\big(1-\rho_+(h(\mathcal{T}))\big)-\partial_h\rho_+(h(\mathcal{T}))re^{-r\mathcal{T}}\log(\mathfrak{b}_0^{\epsilon=1}(\mathcal{T}))}{h(\mathcal{T})\rho_+(h(\mathcal{T}))}\\ &\qquad-\frac{re^{-r\mathcal{T}}\big(\rho_+(h(\mathcal{T}))+h(\mathcal{T})\partial_h\rho_+(h(\mathcal{T}))\big)}{\big(h(\mathcal{T})\rho_+(h(\mathcal{T}))\big)^2}\Bigg]\Big(1-\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)\Big)\\ &\quad-\frac{\Big(\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\Big)^{1-\rho_+(h(\mathcal{T}))}}{re^{-r\mathcal{T}}h(\mathcal{T})\rho_+(h(\mathcal{T}))}\Big(\partial_\mathcal{T}\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)+\partial_x^2\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\Big),\end{aligned}$$

where $\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})$ satisfies the following equation:

$$\begin{aligned}\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T}) &= \partial_\mathcal{T}\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)+\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\\ &\quad+\frac{\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\rho_+(h(\mathcal{T}))-\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\partial_h\rho_+(h(\mathcal{T}))re^{-r\mathcal{T}}}{\rho_+(h(\mathcal{T}))^2}\Big(1-\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)\Big)\\ &\quad-\frac{\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})}{\rho_+(h(\mathcal{T}))}\Big(\partial_\mathcal{T}\partial_x\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)+\partial_x^2\mathcal{C}_E(\mathcal{T},\mathfrak{b}_0^{\epsilon=1}(\mathcal{T});K)\partial_\mathcal{T}\mathfrak{b}_0^{\epsilon=1}(\mathcal{T})\Big).\end{aligned}$$

These results can now be used while implementing higher order algorithms ($n\in\mathbb{N}$). In particular, this allows to improve the stability of subsequent derivatives.[9]

[9] Without the use of such equations, subsequent derivatives would depend on the finite difference steps chosen in the computation of $\partial_h c_{0,0}^+(\cdot)$, which clearly lower their stability.

ii) Implementing our approximations as well as the stability results described in i) requires some (analytical) tractability of the European call $\mathcal{C}_E(\cdot)$ (and of its derivatives) under the respective model. To keep this article self-contained, we therefore recall few results for $\mathcal{C}_E(\cdot)$ under Model (2.3.35) in Appendix A (cf. Section 2.7.1).

♦

2.3.3 Extension to Merton's Jump-Diffusion Model

We next combine the ansatz taken in [Ba91] with the ideas discussed previously to extend our N-th order algorithms to Merton's jump-diffusion model (cf. [Me76]). We assume in the rest of this section that J_1 is normally distributed with mean $\mu_{\mathcal{M}}$ and variance $\sigma_{\mathcal{M}}^2$ or, equivalently, that Π_X is given by

$$\lambda f_{J_1}(y)dy = \Pi_X(dy) = \frac{\lambda}{\sqrt{2\pi\sigma_{\mathcal{M}}^2}} \exp\left(-\frac{(y-\mu_{\mathcal{M}})^2}{2\sigma_{\mathcal{M}}^2}\right) dy, \tag{2.3.47}$$

with $\mu_{\mathcal{M}} \in \mathbb{R}$ and $\sigma_{\mathcal{M}} > 0$ and obtain that $\zeta = e^{\mu_{\mathcal{M}} + \frac{1}{2}\sigma_{\mathcal{M}}^2} - 1$ and that the Laplace exponent equals, for any $\theta \in \mathbb{R}$,

$$\Phi_X(\theta) = \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma^2\Big)\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\big(e^{\theta\mu_{\mathcal{M}} + \frac{1}{2}\theta^2\sigma_{\mathcal{M}}^2} - 1\big). \tag{2.3.48}$$

Additionally, we point out that, by the very same arguments as the ones provided in Section 2.3.2.1, the equation $\Phi_X(\theta) = y$ has for any $y > 0$ two solutions, a positive and a negative root. We follow the notation used in the previous sections and denote by $\Phi_X^{-1,+}(y)$ its positive root.

2.3.3.1 Solution of the 0-th Order Problem

As noted earlier, finding an exact solution to the 0-th order problem, i.e. to Equations (2.3.23)-(2.3.26), is not anymore an easy task, as there is no known solution to Equation (2.3.23). Whenever $\mu_{\mathcal{M}}$ and $\sigma_{\mathcal{M}}$ are "sufficiently small", it seems however reasonable to follow our previous considerations and to use the following approximate solution

$$f_0(h(\mathcal{T}), x; K) = c_{0,0}^{+}(h(\mathcal{T}))x^{\rho_+(h(\mathcal{T}))}, \qquad x \in (0, \mathfrak{b}_0^{\epsilon=1}(\mathcal{T})),\ \mathcal{T} \in (0, T], \tag{2.3.49}$$

with

$$\rho_+(h(\mathcal{T})) := \Phi_X^{-1,+}\Big(\frac{r}{h(\mathcal{T})}\Big), \qquad \mathcal{T} \in (0, T]. \tag{2.3.50}$$

This subsequently allows us to compute $c_{0,0}^{+}(\cdot)$ and $\mathfrak{b}_0^{\epsilon=1}(\cdot)$ via the same approach as the one used in Section 2.3.2.1 and to arrive at Equations (2.3.39), (2.3.40), recovering so the 0-th order premium $f_0(\cdot)$.

2.3.3.2 Solution of the Higher Order Problems

Solving the higher order problems can be done via the same method as the one introduced in Section 2.3.2.2. Indeed, assuming that the n-th order premium $f_n(\cdot)$ has the functional form described by (2.3.41), allows us to derive the following system of $2n$ equations in the $2n$ unknowns $\big(c_{n,j}^{+}(h(\mathcal{T}))\big)_{j\in\{1,\dots,2n\}}$:

$$\begin{aligned} 2n\bigg[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T})) - 1\big) + r - \delta + \lambda(\mathfrak{I}_1(h(\mathcal{T})) - \zeta)\bigg] c_{n,2n}^{+}(h(\mathcal{T})) \\ = r(1 - h(\mathcal{T}))\partial_h\rho_+(h(\mathcal{T}))c_{n-1,2(n-1)}^{+}(h(\mathcal{T})), \end{aligned} \tag{2.3.51}$$

$$j\left[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T}))-1\big)+r-\delta+\lambda(\mathfrak{I}_1(h(\mathcal{T}))-\zeta)\right]c^+_{n,j}(h(\mathcal{T}))$$
$$+j(j+1)\frac{\sigma^2}{2}c^+_{n,j+1}(h(\mathcal{T}))+\lambda\sum_{k=j+1}^{2n}\binom{k}{j-1}\mathfrak{I}_{k-(j-1)}(h(\mathcal{T}))c^+_{n,k}(h(\mathcal{T}))$$
$$=r(1-h(\mathcal{T}))\big(\partial_h c^+_{n-1,j-1}(h(\mathcal{T}))+\partial_h\rho_+(h(\mathcal{T}))c^+_{n-1,j-2}(h(\mathcal{T}))\big), \tag{2.3.52}$$

$$\left[\frac{\sigma^2}{2}\big(2\rho_+(h(\mathcal{T}))-1\big)+r-\delta+\lambda\big(\mathfrak{I}_1(h(\mathcal{T}))-\zeta\big)\right]c^+_{n,1}(h(\mathcal{T}))$$
$$+\sigma^2c^+_{n,2}(h(\mathcal{T}))+\lambda\sum_{k=2}^{2n}\mathfrak{I}_k(h(\mathcal{T}))c^+_{n,k}(h(\mathcal{T}))=r(1-h(\mathcal{T}))\partial_h c^+_{n-1,0}(h(\mathcal{T})), \tag{2.3.53}$$

where Equation (2.3.52) only holds for $j\in\{2,3,\dots,2n-1\}$ and $\mathfrak{I}_\ell(\cdot)$ is given, for any $\ell\in\mathbb{N}$, by

$$\mathfrak{I}_\ell(h(\mathcal{T})):=\int_{\mathbb{R}} y^\ell e^{\rho_+(h(\mathcal{T}))y}f_{J_1}(y)dy. \tag{2.3.54}$$

Using standard calculus, the latter integral can be re-expressed as

$$\mathfrak{I}_\ell(h(\mathcal{T}))=\exp\left\{\rho_+(h(\mathcal{T}))\Big(\mu_{\mathcal{M}}+\frac{\sigma^2_{\mathcal{M}}}{2}\rho_+(h(\mathcal{T}))\Big)\right\}\cdot\sum_{k=0}^{\ell}\binom{\ell}{k}\mu_{\mathcal{M}}^{\ell-k}\,\mathfrak{M}\big(k,\sigma^2_{\mathcal{M}}\rho_+(h(\mathcal{T})),\sigma^2_{\mathcal{M}}\big), \tag{2.3.55}$$

where

$$\mathfrak{M}(k,m,s^2):=\int_{\mathbb{R}} z^k\frac{1}{\sqrt{2\pi s^2}}e^{-\frac{(z-m)^2}{2s^2}}dz \tag{2.3.56}$$

denotes the k-th order non-central moment of the normal distribution having mean m and variance s^2. Therefore, combining (2.3.51)-(2.3.53) with (2.3.55) and (2.3.56) allows us to recover $\big(c^+_{n,j}(h(\mathcal{T}))\big)_{j\in\{1,\dots,2n\}}$ for any $\mathcal{T}\in(0,T]$ via standard numerical techniques. To derive an expression for $c^+_{n,0}(\cdot)$ and $\mathfrak{b}_n^{\epsilon=1}(\cdot)$ we follow the steps outlined in Section 2.3.2.2. This finally leads to Equations (2.3.45) and (2.3.46), from which the n-th order premium $f_n(\cdot)$ is ultimately recovered.

Remark 2.2.

i) As in the model of constant jumps, one can derive (non-linear) equations that help stabilizing higher order approximations. This can be done using Equations (2.3.39)-(2.3.40), (2.3.45)-(2.3.46) and (2.3.50), (2.3.51)-(2.3.53).

ii) Implementing our approximations as well as the stability results described in i) requires some (analytical) tractability of the European call $\mathcal{C}_E(\cdot)$ (and of its derivatives) under the respective model. In Appendix B (cf. Section 2.7.2), we therefore recall few results for $\mathcal{C}_E(\cdot)$ under Model (2.3.47).

iii) Although this article does not investigate jump-diffusion models behind the model of Merton, we believe that the general ideas underlying our method can be combined with the results obtained in [KW04] and [CS14] to derive N-th order approximations to the pricing of American-type options within the whole class of hyper-exponential jump-diffusions. This could be investigated as part of future research.

♦

2.4 Approximation of American Barrier Options

We next adapt the previous method to deal with American barrier options. As an extension of the Barone-Adesi & Whaley algorithm, our ansatz relies once again on the (analytical) tractability of the corresponding European-type options (and Greeks). However, since analytical results for European barrier-type options are mainly known in the setting of Black & Scholes, we focus in the sequel on this model, i.e. we assume from now on that $(S_t)_{t\geq 0}$ evolves according to (2.3.35) with $\varphi = 0$. Investigating the applicability of our method under other asset dynamics (e.g. under certain jump-diffusion dynamics) could be part of future work. Additionally, our derivations will focus on the American down-and-out call (DOC). Nevertheless, we note that our method can be slightly adapted to deal with any other type of (single) barrier options.[10] To illustrate this point, the numerical discussion in Section 2.5 also provides simulation results for the American up-and-out put (UOP).

2.4.1 Pricing Problem and Perturbation Expansion

2.4.1.1 Pricing with Rebates

Let us start by reviewing well-known facts on American down-and-out call options. To keep our derivations applicable in a wide range of problems, we consider barrier options with strike-and-barrier-dependent rebates, i.e. we consider the following American-type down-and-out call option having maturity $\mathcal{T} \geq 0$, initial value $S_0 = x \geq 0$, strike price $K \geq 0$, (lower) barrier level $L \geq 0$ and rebate $\mathcal{R}(K, L)$:

$$\begin{aligned}\mathcal{DOC}_A(\mathcal{T}, x; K, L, \mathcal{R}) &:= \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}_x^{\mathbb{Q}}\left[B_\tau(r)^{-1}\left(S_\tau - K\right)^+ \mathbb{1}_{\{\tau_L > \tau\}} + B_{\tau_L}(r)^{-1}\mathcal{R}(K, L)\mathbb{1}_{\{\tau_L \leq \tau\}}\right] \\ &= \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \Big(\mathbb{E}_x^{\mathbb{Q}}\left[B_\tau(r)^{-1}\left(S_\tau - K\right)^+ \mathbb{1}_{\{\tau_L > \tau\}}\right] + \mathcal{R}(K, L)\mathbb{E}_x^{\mathbb{Q}}\left[B_{\tau_L}(r)^{-1}\mathbb{1}_{\{\tau_L \leq \tau\}}\right] \Big).\end{aligned} \tag{2.4.1}$$

Here $\tau_L := \inf\{t > 0 : \; S_t \leq L\}$ denotes the first passage time of the process $(S_t)_{t\geq 0}$ below the (lower) barrier level L, while $\mathfrak{T}_{[0,\mathcal{T}]}$ refers, as earlier, to the set of stopping times that take values in the interval $[0, \mathcal{T}]$. Additionally, we define the European counterpart to (2.4.1) via

$$\mathcal{DOC}_E(\mathcal{T}, x; K, L, \mathcal{R}) := \mathbb{E}_x^{\mathbb{Q}}\left[B_{\mathcal{T}}(r)^{-1}\left(S_{\mathcal{T}} - K\right)^+ \mathbb{1}_{\{\tau_L > \mathcal{T}\}}\right] + \mathcal{R}(K, L)\mathbb{E}_x^{\mathbb{Q}}\left[B_{\tau_L}(r)^{-1}\mathbb{1}_{\{\tau_L \leq \mathcal{T}\}}\right], \tag{2.4.2}$$

and note that, in the above definitions (2.4.1) and (2.4.2), the rebates are implicitly understood to be paid immediately.

As for standard American options, decomposition techniques are popular methods to price American barrier options. Following this ansatz as well as the line of arguments provided in Section 2.3.1, we therefore define the down-and-out early exercise premium, $\mathcal{E}_{\mathcal{DOC}}(\cdot)$, via

$$\mathcal{E}_{\mathcal{DOC}}(\mathcal{T}, x; K, L, \mathcal{R}) := \mathcal{DOC}_A(\mathcal{T}, x; K, L, \mathcal{R}) - \mathcal{DOC}_E(\mathcal{T}, x; K, L, \mathcal{R}), \tag{2.4.3}$$

and focus on the respective pricing problem for (2.4.3). Here, we first note that the American-type option (2.4.1) should not be exercised before maturity whenever $\delta \leq 0$ and consequently reduces in this case to its European counterpart (2.4.2).[11] Hence, we focus in the sequel on the pricing problem in the non-trivial

[10] More details on barrier options, their relations, and on how to adapt our method to deal with other types of barriers can be found in [JYC06], [GHS00] and [CKKK07].

[11] This is in line with the analysis provided in [GHS00].

case, $\delta > 0$. In this case, one obtains, by slightly adapting the arguments presented in [Ma20],[12] that the pair $\big(\mathcal{E}_{\mathcal{DOC}}(\cdot), \mathfrak{b}_{\mathcal{DOC}}(\cdot)\big)$, where $\mathfrak{b}_{\mathcal{DOC}}(\cdot)$ denotes the down-and-out early exercise boundary, is a solution to the following free-boundary problem:

$$-\partial_{\mathcal{T}}\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},x;K,L,\mathcal{R})+\mathcal{A}_S\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},x;K,L,\mathcal{R})-r\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},x;K,L,\mathcal{R}) = 0, \quad x \in (0,\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T})),\ \mathcal{T} \in (0,T], \tag{2.4.4}$$

subject to the boundary conditions

$$\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}) = \mathfrak{b}_{\mathcal{DOC}}(\mathcal{T}) - K - \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}), \quad \mathcal{T} \in (0,T], \tag{2.4.5}$$

$$\partial_x\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}) = 1 - \partial_x\mathcal{DOC}_E(\mathcal{T},\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}), \quad \mathcal{T} \in (0,T], \tag{2.4.6}$$

$$\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},x;K,L,\mathcal{R}) = 0, \quad x \in [0,L],\ \mathcal{T} \in (0,T], \tag{2.4.7}$$

and initial condition

$$\mathcal{E}_{\mathcal{DOC}}(0,x;K,L,\mathcal{R}) = 0, \quad x \in (0,\mathfrak{b}_{\mathcal{DOC}}(\mathcal{T})). \tag{2.4.8}$$

2.4.1.2 Perturbation Ansatz

We next repeat the ansatz adopted by Barone-Adesi & Whaley (cf. [BW87]) and assume that the early exercise premium $\mathcal{E}_{\mathcal{DOC}}(\cdot)$ takes the form

$$\mathcal{E}_{\mathcal{DOC}}(\mathcal{T},x;K,L,\mathcal{R}) = h(\mathcal{T})F_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}). \tag{2.4.9}$$

This allows us (to transform (2.4.4)-(2.4.7) into an analogue of (2.3.10)-(2.3.13) and) to arrive at the following alternative perturbation problem:

$$-\frac{r}{h(\mathcal{T})}F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) + \mathcal{A}_S F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) - \epsilon r\big(1-h(\mathcal{T})\big)\partial_h F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \tag{2.4.10}$$

on $(\mathcal{T},x) \in (0,T] \times (0,\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T}))$ and with boundary conditions

$$F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}) = \frac{1}{h(\mathcal{T})}\Big(\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T}) - K - \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R})\Big), \quad \mathcal{T} \in (0,T], \tag{2.4.11}$$

$$\partial_x F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R}) = \frac{1}{h(\mathcal{T})}\Big(1 - \partial_x\mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T});K,L,\mathcal{R})\Big), \quad \mathcal{T} \in (0,T], \tag{2.4.12}$$

$$F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \quad x \in [0,L],\ \mathcal{T} \in (0,T]. \tag{2.4.13}$$

As earlier, we assume the existence of functions $\big(f^{\mathcal{DOC}}_n(\cdot)\big)_{n\geq 0}$ and $\big(b^{\mathcal{DOC}}_n(\cdot)\big)_{n\geq 0}$ such that the solution pair $\big(F^{\epsilon}_{\mathcal{DOC}}(\cdot), \mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\cdot)\big)$ to (2.4.10)-(2.4.13) has a representation as "well-behaved" series expansion of the form

$$F^{\epsilon}_{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = \sum_{n=0}^{\infty}\epsilon^n f^{\mathcal{DOC}}_n(h(\mathcal{T}),x;K,L,\mathcal{R}), \quad x \in (0,\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T})), \mathcal{T} \in [0,T], \tag{2.4.14}$$

$$\mathfrak{b}^{\epsilon}_{\mathcal{DOC}}(\mathcal{T}) = \sum_{n=0}^{\infty}\epsilon^n b^{\mathcal{DOC}}_n(\mathcal{T}), \quad \mathcal{T} \in [0,T], \tag{2.4.15}$$

[12] See also [GHS00], [Ga07], and [Al14] for corresponding results.

and define corresponding partial sums of N-th order via

$$F^{\epsilon}_{\mathcal{DOC},N}(h(\mathcal{T}),x;K,L,\mathcal{R}) = \sum_{n=0}^{N} \epsilon^n f_n^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}), \qquad x \in (0,\mathfrak{b}^{\epsilon}_{\mathcal{DOC},N}(\mathcal{T})), \mathcal{T} \in [0,T], \tag{2.4.16}$$

$$\mathfrak{b}^{\epsilon}_{\mathcal{DOC},N}(\mathcal{T}) = \sum_{n=0}^{N} \epsilon^n b_n^{\mathcal{DOC}}(\mathcal{T}), \qquad \mathcal{T} \in [0,T]. \tag{2.4.17}$$

Hence, arguing again as in Section 2.3.1.2 leads us to n-th order analogues to Problems (2.3.23)-(2.3.26) and (2.3.27)-(2.3.30): For $n=0$, the 0-th order problem reads

$$-\frac{r}{h(\mathcal{T})} f_0^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) + \mathcal{A}_S f_0^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \qquad x \in (0,\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T})), \mathcal{T} \in (0,T], \tag{2.4.18}$$

with boundary conditions

$$f_0^{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T});K,L,\mathcal{R}) = \frac{1}{h(\mathcal{T})}\Big(\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T}) - K - \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T});K,L,\mathcal{R})\Big), \qquad \mathcal{T} \in (0,T], \tag{2.4.19}$$

$$\partial_x f_0^{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T});K,L,\mathcal{R}) = \frac{1}{h(\mathcal{T})}\Big(1 - \partial_x \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC},0}(\mathcal{T});K,L,\mathcal{R})\Big), \qquad \mathcal{T} \in (0,T], \tag{2.4.20}$$

$$f_0(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \qquad x \in [0,L], \ \mathcal{T} \in (0,T]. \tag{2.4.21}$$

Additionally, the following higher order problems ($n \in \mathbb{N}$) are obtained:

$$-\frac{r}{h(\mathcal{T})} f_n^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) + \mathcal{A}_S f_n^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) - r\big(1-h(\mathcal{T})\big)\partial_h f_{n-1}^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \tag{2.4.22}$$

on $(\mathcal{T},x) \in (0,T] \times (0,\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T}))$ and with boundary conditions, for $\mathcal{T} \in (0,T]$:

$$\begin{aligned} f_n^{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R}) = \frac{\epsilon^{-n}}{h(\mathcal{T})}\Big(&\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T}) - K - \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R}) \\ &- h(\mathcal{T})F^{\epsilon}_{\mathcal{DOC},n-1}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R})\Big), \end{aligned} \tag{2.4.23}$$

$$\begin{aligned} \partial_x f_n^{\mathcal{DOC}}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R}) = \frac{\epsilon^{-n}}{h(\mathcal{T})}\Big(&1 - \partial_x \mathcal{DOC}_E(\mathcal{T},\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R}) \\ &- h(\mathcal{T})\partial_x F^{\epsilon}_{\mathcal{DOC},n-1}(h(\mathcal{T}),\mathfrak{b}^{\epsilon}_{\mathcal{DOC},n}(\mathcal{T});K,L,\mathcal{R})\Big), \end{aligned} \tag{2.4.24}$$

$$f_n^{\mathcal{DOC}}(h(\mathcal{T}),x;K,L,\mathcal{R}) = 0, \qquad x \in [0,L]. \tag{2.4.25}$$

Solving these problems for $\epsilon = 1$ clearly allows us to recover N-th order approximations of the down-and-out early exercise premium via

$$\mathcal{E}^{\epsilon=1}_{\mathcal{DOC},N}(\mathcal{T},x;K,L,\mathcal{R}) := h(\mathcal{T})F^{\epsilon=1}_{\mathcal{DOC},N}(h(\mathcal{T}),x;K,L,\mathcal{R}) \qquad x \in [0,\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},N}(\mathcal{T})), \tag{2.4.26}$$

$$\mathcal{E}^{\epsilon=1}_{\mathcal{DOC},N}(\mathcal{T},x;K,L,\mathcal{R}) := x - K - \mathcal{DOC}_E(\mathcal{T},x;K,L,\mathcal{R}), \qquad x \in [\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},N}(\mathcal{T}),\infty), \tag{2.4.27}$$

where $\mathcal{T} \in [0,T]$. Therefore, we focus in the sequel on the corresponding problems for $\epsilon = 1$.

2.4.2 Derivation of the Solutions

2.4.2.1 Solution of the 0-th Order Problem

To derive a solution to the 0-th order problem, we decompose the (state-)domain of Equation (2.4.18) for any $\mathcal{T} \in (0, T]$ into two intervals, $I_0 := [0, L]$ and $I_1 := (L, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T}))$, derive solutions $V_0^{\mathcal{DOC}}(\cdot)$, $V_1^{\mathcal{DOC}}(\cdot)$ on these domains and combine them to recover $f_0^{\mathcal{DOC}}(\cdot)$ via

$$f_0^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) = \begin{cases} V_0^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}), & x \in I_0, \\ V_1^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}), & x \in I_1. \end{cases} \tag{2.4.28}$$

First, it is clear that $V_0^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) \equiv 0$ must hold for $x \in I_0$. Therefore, we only need to derive an expression for $V_1^{\mathcal{DOC}}(\cdot)$. Here, following the arguments provided in Section 2.3.2.1, the general solution of the homogeneous equation (2.4.18) on I_1 is obtained as

$$V_1^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) = c_{0,0}^{\mathcal{DOC},+}(h(\mathcal{T}))x^{\rho_+(h(\mathcal{T}))} + c_{0,0}^{\mathcal{DOC},-}(h(\mathcal{T}))x^{\rho_-(h(\mathcal{T}))}, \qquad x \in I_1,\ \mathcal{T} \in (0, T], \tag{2.4.29}$$

where $\rho_+(\cdot)$ and $\rho_-(\cdot)$ are defined as in (2.3.38) but with $\varphi = 0$ and $c_{0,0}^{\mathcal{DOC},+}(\cdot)$, $c_{0,0}^{\mathcal{DOC},-}(\cdot)$ are "constants" to be determined. To conclude, we therefore need to determine $c_{0,0}^{\mathcal{DOC},+}(\cdot)$, $c_{0,0}^{\mathcal{DOC},-}(\cdot)$ as well as $\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\cdot)$. This is done by combining Conditions (2.4.19)-(2.4.21). Indeed, Condition (2.4.21) first implies that

$$c_{0,0}^{\mathcal{DOC},-}(h(\mathcal{T})) = -L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))} \cdot c_{0,0}^{\mathcal{DOC},+}(h(\mathcal{T})), \qquad \mathcal{T} \in (0, T]. \tag{2.4.30}$$

Then, combining (2.4.30) with Condition (2.4.20) allows us to derive, for $\mathcal{T} \in (0, T]$, that

$$c_{0,0}^{\mathcal{DOC},+}(h(\mathcal{T})) = \frac{1 - \partial_x \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T}); K, L, \mathcal{R})}{h(\mathcal{T})\Big(\rho_+(h(\mathcal{T}))\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_+(h(\mathcal{T}))-1} - \rho_-(h(\mathcal{T}))L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))}\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_-(h(\mathcal{T}))-1}\Big)} \tag{2.4.31}$$

and inserting the latter expression into (2.4.19) finally gives that $\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})$ solves, for any $\mathcal{T} \in (0, T]$, the following non-linear equation

$$\begin{aligned} \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T}) &= K + \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T}); K, L, \mathcal{R}) \\ &+ \frac{\Big(1 - \partial_x \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T}); K, L, \mathcal{R})\Big)\Big(\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_+(h(\mathcal{T}))} - L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))}\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_-(h(\mathcal{T}))}\Big)}{\rho_+(h(\mathcal{T}))\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_+(h(\mathcal{T}))-1} - \rho_-(h(\mathcal{T}))L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))}\big(\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})\big)^{\rho_-(h(\mathcal{T}))-1}}. \end{aligned} \tag{2.4.32}$$

Therefore, solving for any $\mathcal{T} \in (0, T]$ Equation (2.4.32) for $\mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},0}(\mathcal{T})$ numerically allows us to recover $c_{0,0}^{\mathcal{DOC},+}(h(\mathcal{T}))$, $c_{0,0}^{\mathcal{DOC},-}(h(\mathcal{T}))$, and subsequently $f_0^{\mathcal{DOC}}(\cdot)$ via (2.4.28).

2.4.2.2 Solution of the Higher Order Problems

We finally seek, for $n \in \mathbb{N}$, a solution to Problem (2.4.22)-(2.4.25). As in the previous section, we define $I_0 := [0, L]$ and $I_1 := (L, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},n}(\mathcal{T}))$, decompose $f_n^{\mathcal{DOC}}(\cdot)$, for any $\mathcal{T} \in (0, T]$, via

$$f_n^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) = \begin{cases} V_{n,0}^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}), & x \in I_0, \\ V_{n,1}^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}), & x \in I_1, \end{cases} \tag{2.4.33}$$

and derive expressions for the relevant functions. In view of Equation (2.4.25), it directly follows that $V_{n,0}^{\mathcal{DOC}}(h(\mathcal{T}), x; K) \equiv 0$ must hold on I_0. Furthermore, following the ansatz taken in Section 2.3.2.2, we now assume that $V_{n,1}^{\mathcal{DOC}}(\cdot)$ takes for $x \in (0, \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}))$, $\mathcal{T} \in (0, T]$, the following form

$$\begin{aligned} V_{n,1}^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) = &\left(c_{n,0}^{\mathcal{DOC},+}(h(\mathcal{T})) + \sum_{j=1}^{2n} c_{n,j}^{\mathcal{DOC},+}(h(\mathcal{T})) \log(x)^j \right) x^{\rho_+(h(\mathcal{T}))} \\ &+ \left(c_{n,0}^{\mathcal{DOC},-}(h(\mathcal{T})) + \sum_{j=1}^{2n} c_{n,j}^{\mathcal{DOC},-}(h(\mathcal{T})) \log(x)^j \right) x^{\rho_-(h(\mathcal{T}))}, \end{aligned} \tag{2.4.34}$$

and derive a system of equations in the coefficients $\left(c_{n,j}^{\mathcal{DOC},\pm}(\cdot)\right)_{j\in\{1,\dots,2n\}}$. Indeed, proceeding as in Section 2.3.2.2 gives that the coefficients $\left(c_{n,j}^{\mathcal{DOC},\pm}(h(\mathcal{T}))\right)_{j\in\{1,\dots,2n\}}$ solve for any $\mathcal{T} \in (0, T]$ System (2.3.42)-(2.3.44), where $\rho_+(h(\mathcal{T})$ is then replaced by $\rho_\pm(h(\mathcal{T}))$ and $\varphi = 0$. To conclude, we therefore need to determine $c_{n,0}^{\mathcal{DOC},+}(\cdot)$, $c_{n,0}^{\mathcal{DOC},-}(\cdot)$ and $\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\cdot)$ and this is done via the same methods as the ones used in the previous section: First, we obtain from Condition (2.4.25) that

$$c_{n,0}^{\mathcal{DOC},-}(h(\mathcal{T})) = -\left(L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))} \cdot c_{0,0}^{\mathcal{DOC},+}(h(\mathcal{T})) + \mathfrak{R}_n^\star(h(\mathcal{T}), L; K, L, \mathcal{R}) \right), \qquad \mathcal{T} \in (0, T], \tag{2.4.35}$$

where, for $x \in [0, \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T})]$ and $\mathcal{T} \in (0, T]$, the "rest term", $\mathfrak{R}_n^\star(\cdot)$, equals

$$\mathfrak{R}_n^\star(h(\mathcal{T}), x; K, L, \mathcal{R}) := L^{-\rho_-(h(\mathcal{T}))} \cdot V_{n,1}^{\mathcal{DOC},\star}(h(\mathcal{T}), x; K, L, \mathcal{R}), \tag{2.4.36}$$

and $V_{n,1}^{\mathcal{DOC},\star}(\cdot)$ is defined via

$$V_{n,1}^{\mathcal{DOC},\star}(h(\mathcal{T}), x; K, L, \mathcal{R}) := V_{n,1}^{\mathcal{DOC}}(h(\mathcal{T}), x; K, L, \mathcal{R}) - \left(c_{n,0}^{\mathcal{DOC},+}(h(\mathcal{T})) x^{\rho_+(h(\mathcal{T}))} + c_{n,0}^{\mathcal{DOC},-}(h(\mathcal{T})) x^{\rho_-(h(\mathcal{T}))} \right). \tag{2.4.37}$$

Then, rewriting (2.4.24) using Representation (2.4.35) leads to

$$\begin{aligned} c_{n,0}^{\mathcal{DOC},+}(h(\mathcal{T})) = &\frac{1 - \partial_x \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) - h(\mathcal{T}) \partial_x F_{\mathcal{DOC},n-1}^{\epsilon=1}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R})}{h(\mathcal{T}) \left(\rho_+(h(\mathcal{T})) \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_+(h(\mathcal{T}))-1} - \rho_-(h(\mathcal{T})) L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))} \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_-(h(\mathcal{T}))-1} \right)} \\ &- \frac{\partial_x V_{n,1}^{\mathcal{DOC},\star}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) - \rho_-(h(\mathcal{T})) \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_-(h(\mathcal{T}))-1} \mathfrak{R}_n^\star(h(\mathcal{T}), L; K, L, \mathcal{R})}{\rho_+(h(\mathcal{T})) \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_+(h(\mathcal{T}))-1} - \rho_-(h(\mathcal{T})) L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))} \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_-(h(\mathcal{T}))-1}} \end{aligned} \tag{2.4.38}$$

and inserting the latter expression into (2.4.23) finally gives that $\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(h(\mathcal{T}))$ solves, for any $\mathcal{T} \in (0, T]$, the following non-linear equation

$$\begin{aligned} \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) = K &+ \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) + h(\mathcal{T}) F_{\mathcal{DOC},n-1}^{\epsilon=1}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) \\ &+ h(\mathcal{T}) V_{n,1}^{\mathcal{DOC},\star}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) - h(\mathcal{T}) \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_-(h(\mathcal{T}))} \mathfrak{R}_n^\star(h(\mathcal{T}), L; K, L, \mathcal{R}) \\ &+ \mathcal{Q}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T})) \Big[1 - \partial_x \mathcal{DOC}_E(\mathcal{T}, \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K) \\ &\quad - h(\mathcal{T}) \Big(\partial_x F_{\mathcal{DOC},n-1}^{\epsilon=1}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) + \partial_x V_{n,1}^{\mathcal{DOC},\star}(h(\mathcal{T}), \mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}); K, L, \mathcal{R}) \\ &\qquad - \rho_-(h(\mathcal{T})) \big(\mathfrak{b}_{\mathcal{DOC},n}^{\epsilon=1}(\mathcal{T}) \big)^{\rho_-(h(\mathcal{T}))-1} \mathfrak{R}_n^\star(h(\mathcal{T}), L; K, L, \mathcal{R}) \Big) \Big] \end{aligned} \tag{2.4.39}$$

with

$$\mathcal{Q}(h(\mathcal{T}),x) := \frac{x^{\rho_+(h(\mathcal{T}))} - L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))}x^{\rho_-(h(\mathcal{T}))}}{\rho_+(h(\mathcal{T}))x^{\rho_+(h(\mathcal{T}))-1} - \rho_-(h(\mathcal{T}))L^{\rho_+(h(\mathcal{T}))-\rho_-(h(\mathcal{T}))}x^{\rho_-(h(\mathcal{T}))-1}}, \qquad x \in [0, \mathfrak{b}^{\epsilon=1}_{\mathcal{DOC},n}(\mathcal{T})]. \tag{2.4.40}$$

Therefore, using Equations (2.4.39), (2.4.38) and (2.4.35), we can deduce all the remaining unknowns and recover $f_n^{\mathcal{DOC}}(\cdot)$ via (2.4.33).

2.5 Numerical Results

In this section, our approximations of up to order three are tested via numerical experiments. We combine a variety of parameters that were used in similar simulation studies provided in [BW87], [Ba91], [GHS00], [AIL03], [JC04], [CKKK07] and [FMRZ15]. Although the resulting parameter constellations do not reflect the current market situation, testing option pricing problems with these parameters allows for a direct comparison of the results across articles and has therefore become a standard over the years. For this reason we also stick with these parameters here. We discuss the accuracy and efficiency of our approximations via classical methods. In particular, we use the root mean squared error (RMSE) as measure of accuracy, while the total CPU time (in seconds) required to execute the algorithms is considered as measure of efficiency. All our numerical experiments are obtained using Matlab R2017b on an Intel CORE i7 processor.

Table 2.1: Theoretical call values for $K = 100$, $r - \delta = -0.04$, $\lambda = 2.5$, $\mu_{\mathcal{M}} = 0.05$, $\sigma_{\mathcal{M}} = 0.03$.

Call Option Prices													
		Model of Constant Jumps						*Merton's Jump-Diffusion Model*					
Parameters		*European*	*American*					*European*	*American*				
				N-th Order Approx.						*N-th Order Approx.*			
	S_0	*Europ. Price*	*Bench-mark*	$N=0$	$N=1$	$N=2$	$N=3$	*Europ. Price*	*Bench-mark*	$N=0$	$N=1$	$N=2$	$N=3$
(1)	80	0.061	0.062	0.065	0.057	0.064	0.062	0.084	0.086	0.090	0.081	0.089	0.086
$r = 0.08$	90	0.749	0.764	0.773	0.757	0.766	0.766	0.831	0.849	0.860	0.843	0.852	0.851
$\sigma = 0.2$	100	3.719	3.833	3.831	3.822	3.834	3.835	3.821	3.939	3.941	3.932	3.943	3.944
$\mathcal{T} = 0.25$	110	10.043	10.525	10.483	10.516	10.527	10.527	10.098	10.572	10.541	10.571	10.583	10.583
	120	18.681	20.000	20.000	20.000	20.000	20.000	18.697	20.000	20.000	20.000	20.000	20.000
(2)	80	0.643	0.671	0.704	0.650	0.671	0.680	0.730	0.763	0.799	0.742	0.764	0.772
$r = 0.08$	90	2.262	2.394	2.441	2.368	2.388	2.401	2.411	2.555	2.604	2.530	2.551	2.563
$\sigma = 0.2$	100	5.597	6.035	6.061	6.001	6.023	6.037	5.773	6.225	6.257	6.196	6.219	6.232
$\mathcal{T} = 0.75$	110	10.834	11.972	11.936	11.935	11.959	11.970	10.991	12.126	12.101	12.098	12.123	12.133
	120	17.676	20.149	20.102	20.138	20.148	20.151	17.787	20.201	20.161	20.200	20.212	20.215
(3)	80	1.482	1.623	1.714	1.587	1.601	1.637	1.622	1.779	1.875	1.743	1.757	1.795
$r = 0.08$	90	3.480	3.901	4.009	3.859	3.867	3.906	3.678	4.126	4.239	4.086	4.095	4.135
$\sigma = 0.2$	100	6.693	7.718	7.798	7.667	7.675	7.713	6.924	7.977	8.065	7.931	7.941	7.979
$\mathcal{T} = 1.50$	110	11.147	13.292	13.297	13.236	13.249	13.279	11.379	13.530	13.549	13.482	13.497	13.527
	120	16.704	20.712	20.654	20.675	20.686	20.702	16.913	20.857	20.814	20.830	20.843	20.861
RMSE			–	**0.051**	**0.031**	**0.021**	**0.007**		–	**0.052**	**0.027**	**0.017**	**0.008**
CPU (sec.)			**2306.07**	**0.07**	**0.47**	**1.41**	**3.19**		**2381.86**	**0.07**	**0.48**	**1.42**	**3.21**

Table 2.2: Theoretical call and put values for $K = 100$, $r - \delta = 0.00$, $\lambda = 2.5$, $\mu_{\mathcal{M}} = 0.05$, $\sigma_{\mathcal{M}} = 0.03$.

Call and Put Option Prices under Merton's Jump-Diffusion Model														
		Call Option Prices						*Put Option Prices*						
Parameters		*European*	*American*					*European*	*American*					
				N-th Order Approx.						*N-th Order Approx.*				
	S_0	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$
(1)	80	0.105	0.105	0.106	0.103	0.107	0.103	19.709	20.004	20.000	20.004	20.004	20.003
$r = 0.08$	90	0.982	0.984	0.988	0.981	0.987	0.982	10.784	10.852	10.849	10.847	10.855	10.849
$\sigma = 0.2$	100	4.304	4.323	4.328	4.319	4.327	4.322	4.304	4.315	4.321	4.311	4.319	4.313
$\mathcal{T} = 0.25$	110	10.964	11.049	11.045	11.044	11.053	11.049	1.162	1.164	1.168	1.160	1.167	1.161
	120	19.814	20.074	20.063	20.075	20.080	20.077	0.210	0.210	0.212	0.207	0.214	0.207
(2)	80	1.012	1.020	1.038	1.000	1.034	1.017	19.848	20.495	20.481	20.475	20.500	20.493
$r = 0.08$	90	3.149	3.183	3.213	3.158	3.197	3.181	12.567	12.816	12.840	12.786	12.825	12.815
$\sigma = 0.2$	100	7.171	7.283	7.319	7.254	7.296	7.282	7.171	7.262	7.300	7.232	7.274	7.261
$\mathcal{T} = 0.75$	110	13.114	13.401	13.426	13.370	13.413	13.401	3.696	3.728	3.762	3.699	3.743	3.726
	120	20.571	21.193	21.190	21.166	21.204	21.194	1.736	1.746	1.771	1.719	1.763	1.743
(3)	80	2.499	2.551	2.624	2.490	2.574	2.562	20.237	21.488	21.532	21.432	21.490	21.491
$r = 0.08$	90	5.333	5.479	5.582	5.409	5.498	5.491	14.202	14.820	14.923	14.748	14.826	14.829
$\sigma = 0.2$	100	9.542	9.885	10.003	9.808	9.898	9.896	9.542	9.846	9.969	9.769	9.859	9.860
$\mathcal{T} = 1.50$	110	15.037	15.731	15.841	15.653	15.740	15.740	6.168	6.317	6.434	6.238	6.337	6.334
	120	21.596	22.856	22.931	22.782	22.862	22.862	3.857	3.930	4.030	3.851	3.956	3.947
RMSE			–	**0.058**	**0.045**	**0.012**	**0.006**		–	**0.061**	**0.045**	**0.012**	**0.008**
CPU (sec.)			**2359.24**	**0.08**	**0.49**	**1.58**	**3.51**		**2398.87**	**0.06**	**0.37**	**1.19**	**2.66**

2.5.1 Standard American Options

We start by discussing our approximations for standard American options under the model of constant jumps as well as under Merton's jump-diffusion model (cf. [Me76]). For each set of parameters, our approximations are tested as follows: We first compute the true European value of the option in the respective model and subsequently determine the early exercise premium via an explicit finite difference scheme.[13] Adding this premium to the corresponding European value allows us to build a benchmark for the American option price against which the approximations are finally tested. Compared with a direct application of our explicit scheme to the American option, this decomposition approach has some benefits. In particular, applying the

[13] Our finite difference scheme corresponds to a fully explicit (American) version of the explicit-implicit method presented in [CV05b]. Instead of working with PIDEs in price coordinates, this method is based on the corresponding PIDEs in log-moneyness coordinate. For an American call, this means that we first transform the pricing problem via

$$u(\mathcal{T}, x) := \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \left(e^{x + X_\tau} - 1 \right)^+ \right],$$

$$u(\mathcal{T}, \mathbf{x}) = K \cdot \mathcal{C}_A(\mathcal{T}, x; K), \qquad \mathbf{x} = \log\left(\frac{x}{K}\right),$$

and solve the resulting early exercise problem. Hence, in the continuation region the PIDE considered so far

$$-\partial_{\mathcal{T}} \mathcal{C}_A(\mathcal{T}, x; K) + \mathcal{A}_S \mathcal{C}_A(\mathcal{T}, x; K) - r\mathcal{C}_A(\mathcal{T}, x; K) = 0$$

transforms to the following log-moneyness equation

$$\partial_{\mathcal{T}} u(\mathcal{T}, \mathbf{x}) = \frac{1}{2}\sigma^2 \partial^2_{\mathbf{x}} u(\mathcal{T}, \mathbf{x}) + \left(r - \delta - \lambda\zeta - \frac{1}{2}\sigma^2\right) \partial_{\mathbf{x}} u(\mathcal{T}, \mathbf{x}) + \lambda \int_{\mathbb{R}} \left(u(\mathcal{T}, \mathbf{x} + y) - u(\mathcal{T}, \mathbf{x}) \right) f_{J_1}(y) dy - ru(\mathcal{T}, \mathbf{x})$$

and the corresponding (early-exercise) free-boundary problem is solved using a fully explicit finite difference scheme.

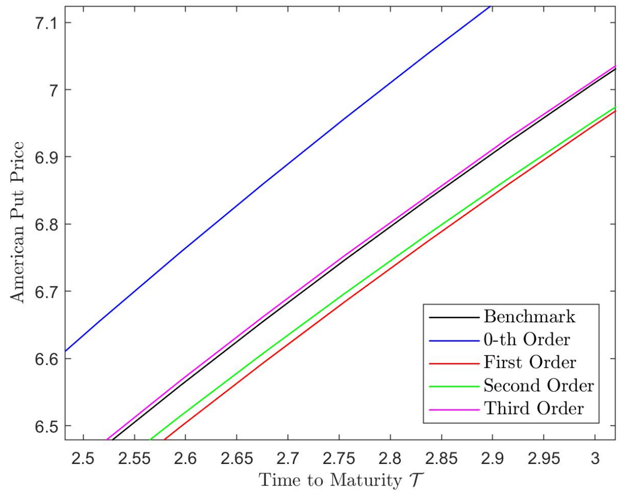
explicit scheme to the early exercise premium instead substantially reduces the pricing errors and therefore leads to more accuracy in our benchmark.

To test our approximations, we combine the choices made in [BW87], [Ba91], [JC04], and [FMRZ15]. For the diffusion as well as the option specific parameters, we rely on [BW87], [FMRZ15] and take $\sigma = 0.2$, $r = 0.08$, $r - \delta =: b \in \{-0.04, 0.00, 0.04\}$, $S_0 \in \{80, 90, 100, 110, 120\}$ and $K = 100$. For the jump parameters, we combine the choices made in [Ba91] and [JC04]: First, we take $\lambda = 2.5$. Although this parameter is neither used in [Ba91] nor in [JC04], it provides a sensible choice between the conservative value of [JC04], $\lambda = 1$, and the more extreme choice in [Ba91], $\lambda = 10$. In any cases, we will see that changing this parameter does not substantially alter the quality of the results obtained in this section (cf. Figure 2.2b). For the volatility of jumps, we rely on the parameters in [Ba91] and fix $\sigma_{\mathcal{M}} = 0.03$. Finally, we consider $\varphi = \mu_{\mathcal{M}} = 0.05$. This choice results for the model of constant jumps in jump sizes of $e^{\varphi} - 1 \approx 0.051$ and for Merton's jump-diffusion model in $\zeta \approx 0.052$. Here again, we note that changing the jump sizes in a sensible range does not alter our results substantially (cf Figure 2.2c). We will further investigate the impact of the volatility level σ, the jump intensity λ, and the jump size $\mu_{\mathcal{M}}$ on the accuracy of our methods at the end of this section. The results are summarized in Tables 2.1-2.3.

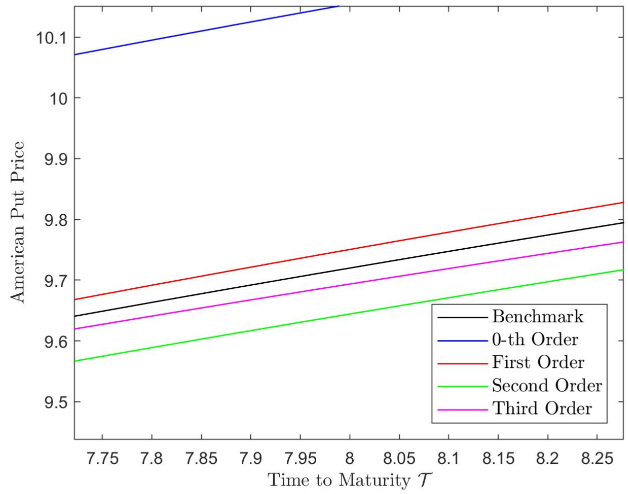
Table 2.3: Theoretical put values for $K = 100$, $r - \delta = 0.04$, $\lambda = 2.5$, $\mu_{\mathcal{M}} = 0.05$, $\sigma_{\mathcal{M}} = 0.03$.

Put Option Prices													
		Model of Constant Jumps						*Merton's Jump-Diffusion Model*					
Parameters		*European*	*American*					*European*	*American*				
				N-th Order Approx.						*N-th Order Approx.*			
	S_0	*Europ. Price*	*Bench-mark*	$N=0$	$N=1$	$N=2$	$N=3$	*Europ. Price*	*Bench-mark*	$N=0$	$N=1$	$N=2$	$N=3$
(1)	80	18.914	20.000	20.000	20.000	20.000	20.000	18.945	20.000	20.000	20.000	20.000	20.000
$r = 0.08$	90	9.977	10.371	10.331	10.365	10.373	10.370	10.069	10.430	10.394	10.425	10.432	10.429
$\sigma = 0.2$	100	3.748	3.832	3.831	3.824	3.833	3.830	3.843	3.917	3.921	3.912	3.920	3.916
$\mathcal{T} = 0.25$	110	0.938	0.950	0.960	0.945	0.953	0.949	0.981	0.992	1.002	0.987	0.994	0.990
	120	0.156	0.158	0.163	0.152	0.160	0.157	0.167	0.168	0.173	0.162	0.171	0.167
(2)	80	17.803	20.000	20.000	20.000	20.000	20.000	17.923	20.008	20.000	20.000	20.008	20.008
$r = 0.08$	90	10.718	11.606	11.562	11.578	11.602	11.606	10.888	11.736	11.699	11.709	11.733	11.736
$\sigma = 0.2$	100	5.754	6.092	6.112	6.065	6.089	6.095	5.922	6.247	6.272	6.221	6.245	6.250
$\mathcal{T} = 0.75$	110	2.771	2.893	2.935	2.869	2.892	2.897	2.898	3.015	3.061	2.992	3.016	3.020
	120	1.211	1.253	1.292	1.230	1.255	1.258	1.289	1.329	1.371	1.306	1.333	1.335
(3)	80	16.883	20.204	20.154	20.187	20.198	20.202	17.082	20.279	20.227	20.260	20.273	20.278
$r = 0.08$	90	11.207	12.840	12.831	12.794	12.822	12.839	11.440	13.027	13.028	12.982	13.011	13.028
$\sigma = 0.2$	100	7.089	7.888	7.957	7.841	7.870	7.892	7.316	8.101	8.178	8.054	8.084	8.107
$\mathcal{T} = 1.50$	110	4.302	4.691	4.795	4.647	4.677	4.701	4.497	4.883	4.992	4.838	4.869	4.894
	120	2.523	2.712	2.817	2.668	2.701	2.726	2.674	2.862	2.972	2.818	2.853	2.878
RMSE			–	**0.049**	**0.027**	**0.008**	**0.005**		–	**0.052**	**0.027**	**0.008**	**0.006**
CPU (sec.)			**2322.10**	**0.06**	**0.39**	**1.18**	**2.63**		**2401.61**	**0.06**	**0.40**	**1.19**	**2.65**

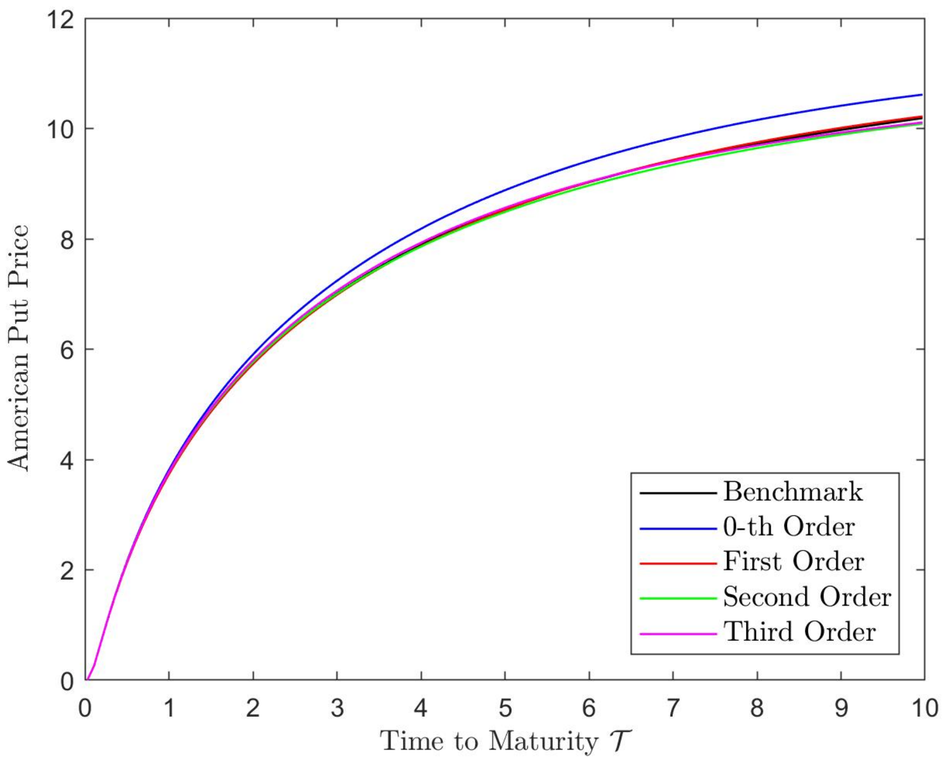
Several facts can be observed from the numerical results reported in Tables 2.1-2.3. First, we observe that a high pricing accuracy can be obtained by increasing the order of our approximations. Indeed, compared to the 0-th order approximation, i.e. Bates' method, any higher order approximation augments the pricing accuracy significantly. In addition, increasing the order of the approximation by one roughly halves the absolute pricing errors (RMSE) made by the method. However, this happens at the expense of greater computational complexity (CPU). Secondly, Table 2.1 and Table 2.3 reveal that all our approximations exhibit a similar behavior in both models, the model of constant jumps and Merton's jump-diffusion model.



(a) Partial graph: $\mathcal{T} \in (2.49, 3.01)$.



(b) Partial graph: $\mathcal{T} \in (7.73, 8.27)$.



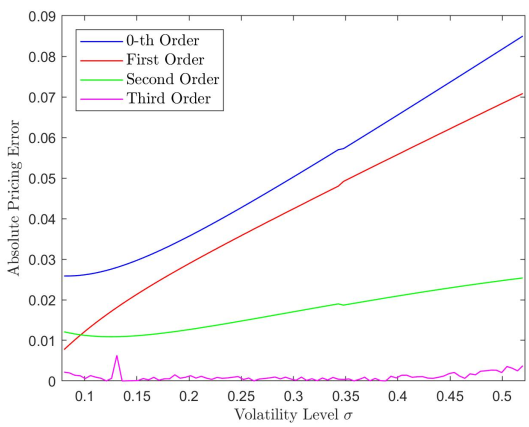
(c) Full graph: $\mathcal{T} \in (0, 10)$.

Figure 2.1: American put price as function of the time to maturity $\mathcal{T} \in (0, 10)$ when the parameters are chosen as: $\sigma = 0.2$, $r = 0.08$, $r - \delta = 0.04$, $\lambda = 2.5$, $\mu_{\mathcal{M}} = 0.05$, $\sigma_{\mathcal{M}} = 0.03$, $S_0 = 110$, $K = 100$.

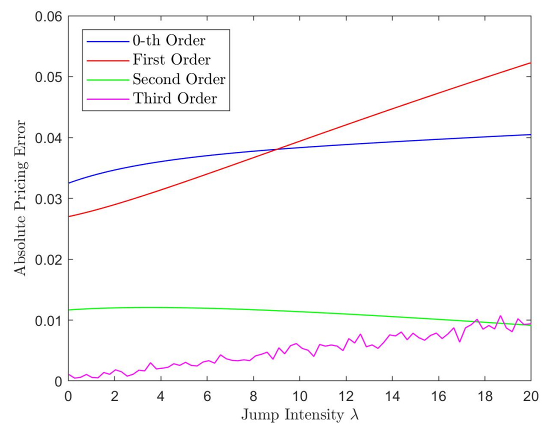
This is not surprising, as the model of constant jumps can be obtained as a limiting case of Merton's jump-diffusion model, namely when $\sigma_{\mathcal{M}} \downarrow 0$. This also justifies our choice to restrict our analysis to call and put options under Merton's jump-diffusion model in Table 2.2 as well as in Figure 2.1 and Figure 2.2. Finally, we should mention that our approximations of higher orders do not outperform the 0-th order method when the early exercise premium becomes very small. This has shown up in numerical simulations.[14] In such cases, however, the European value already provides good results for the American price and relying on this value gives the best approximation.

We next look at the impact of an increase in time to maturity on the accuracy of our approximations. This is exemplified in Figure 2.1, where we have plotted, for $\mathcal{T} \in (0, 10)$ and $r - \delta = 0.04$, out-of-the money American put option prices computed via our explicit finite difference scheme (Benchmark) as well as our corresponding approximations of order up to three. The results are in line with the observations

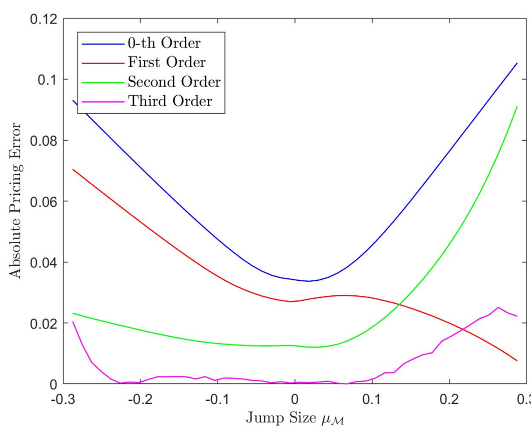
[14] In the case of a call options, this holds for $b = 0.04$, whenever $T \in (0, 2]$ roughly.



(a) Volatility Graph: $\sigma \in (0.075, 0.525)$.



(b) Jump Intensity Graph: $\lambda \in (0, 20)$.



(c) Jump Size Graph: $\mu_{\mathcal{M}} \in (-0.3, 0.3)$.

Figure 2.2: Absolute call option pricing errors as functions of the volatility $\sigma \in (0.075, 0.525)$, the jump intensity $\lambda \in (0, 20)$ and the jump size $\mu_{\mathcal{M}} \in (-0.3, 0.3)$, when the remaining parameters are chosen as: $\sigma = 0.2$, $r = 0.08$, $r - \delta = 0.00$, $\lambda = 2.5$, $\sigma_{\mathcal{M}} = 0.03$, $S_0 = 100$, $K = 100$, $\mathcal{T} = 0.75$.

obtained for Tables 2.1-2.3. Indeed, as in Tables 2.1-2.3, increasing the order of our approximations is shown to substantially augment the accuracy of the method on $\mathcal{T} \in (0, 10)$. In particular, while the 0-th order approximation tends to move substantially away from the benchmark as time increases, higher order versions seem to be more robust and stay impressively close to the "true" value.

Table 2.4: Theoretical down-and-out call values for $K = 45$, $\delta = 0.025$ and barrier level $L = 40$.

Down-and-Out Call Option Prices													
		Volatility Param. $\sigma = 0.2$						*Volatility Param.* $\sigma = 0.4$					
Parameters		*European*	*American*					*European*	*American*				
				N-th Order Approx.						*N-th Order Approx.*			
	S_0	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$
	40.5	0.142	0.142	0.142	0.142	0.142	0.142	0.307	0.307	0.307	0.307	0.307	0.307
(1)	42.5	0.771	0.771	0.771	0.771	0.771	0.771	1.543	1.543	1.543	1.542	1.543	1.543
$r = 4.88\%$	45	1.900	1.900	1.900	1.900	1.900	1.900	3.151	3.151	3.151	3.150	3.151	3.151
$\mathcal{T} = 0.25$	47.5	3.519	3.519	3.519	3.519	3.519	3.519	4.883	4.883	4.883	4.882	4.883	4.883
	50	5.548	5.548	5.548	5.548	5.548	5.547	6.760	6.760	6.760	6.759	6.761	6.760
	40.5	0.307	0.307	0.307	0.307	0.307	0.307	0.411	0.411	0.412	0.411	0.411	0.411
(2)	42.5	1.522	1.522	1.522	1.521	1.523	1.521	2.045	2.046	2.048	2.044	2.046	2.046
$r = 4.88\%$	45	3.100	3.100	3.100	3.099	3.102	3.099	4.080	4.081	4.085	4.078	4.081	4.081
$\mathcal{T} = 0.75$	47.5	4.828	4.828	4.829	4.826	4.831	4.827	6.125	6.126	6.132	6.122	6.126	6.126
	50	6.732	6.732	6.733	6.729	6.737	6.731	8.193	8.195	8.203	8.190	8.195	8.195
	40.5	0.404	0.404	0.404	0.403	0.404	0.404	0.456	0.457	0.458	0.456	0.457	0.457
(3)	42.5	1.976	1.976	1.977	1.972	1.978	1.977	2.264	2.269	2.276	2.266	2.269	2.269
$r = 4.88\%$	45	3.900	3.900	3.904	3.893	3.905	3.903	4.501	4.510	4.524	4.506	4.510	4.510
$\mathcal{T} = 1.50$	47.5	5.839	5.840	5.845	5.829	5.846	5.843	6.723	6.736	6.757	6.730	6.737	6.737
	50	7.829	7.829	7.837	7.815	7.837	7.834	8.937	8.957	8.983	8.948	8.957	8.957
RMSE ($\times 10^{-1}$)			–	**0.028**	**0.052**	**0.033**	**0.018**		–	**0.099**	**0.036**	**0.004**	**0.003**
CPU (sec.)			**1502.14**	**0.011**	**0.036**	**0.083**	**0.171**		**1501.06**	**0.014**	**0.053**	**0.134**	**0.283**

Lastly, we investigate the impact of the volatility level σ, the jump intensity λ, and the jump size $\mu_{\mathcal{M}}$ on the accuracy of our method. To this end, we have plotted, for $r = 0.08$, $r - \delta = 0.00$, $S_0 = 100$, $K = 100$, and time to maturity $\mathcal{T} = 0.75$, the absolute call option pricing errors as functions of the volatility level $\sigma \in (0.075, 0.525)$, the jump intensity $\lambda \in (0, 20)$, and the jump size $\mu_{\mathcal{M}} \in (-0.3, 0.3)$. The graphs are provided in Figure 2.2. Here again, the results are in line with our previous observations. In particular, we see that increasing the order of our approximations leads to an impressive decrease of the pricing error for a very large range of parameters. With respect to the jump size, we note that this holds true for negative jumps as well as for positive jumps roughly up to the size of $\zeta \approx 0.14$. Similarly the results hold true for intensities roughly up to $\lambda = 10$. As seen in Section 2.3.3 (see also Section 2.3.2), we note however that our general solution ansatz is expected to deviate from the true solution, for call options, whenever positive jumps have a considerable impact on the asset dynamics. This is in particular the case when either "large" positive jumps or "large" jump intensities are considered. This possibly explains the loss of monotonicity in the pricing accuracy of our approximations observed in Figure 2.2b and Figure 2.2c. In any cases, we observe that all our higher order approximations substantially beat the 0-th order version for a sensible range of parameters and that our approximation of order three exhibits a remarkable accuracy on the full set of parameters tested.

2.5.2 American Barrier Options

We now turn to a discussion of our approximations for American barrier options under the model of Black & Scholes. For each set of parameters, our approximations are tested against Ritchken's trinomial tree method with 5000 time steps. A similar benchmark was used in [CKKK07], where the authors used 10000 time steps instead. However, we note that choosing 5000 times steps does not alter the results for all the parameter sets considered here. Following the simulations offered in [CKKK07], we restrict our tests to regular down-and-out call options as well as to regular and reverse up-and-out put options. However, we note that considering other barrier types should not alter the quality of our results, as this merely requires simple adaptions.

We start by considering regular down-and-out call options and regular up-and-out put options. To allow for a direct comparability of our results with the existing literature, we mainly rely on the parameters used in [GHS00] and [CKKK07], i.e. we take $\sigma \in \{0.2, 0.4\}$, $r = 0.0488$, $\delta = 0.025$ and $K = 45$. For down-and-out call options we choose additionally $S_0 \in \{40.5, 42.5, 45, 47.5, 50\}$ and barrier level $L = 40$ while these parameters are "reversed" in the case of up-and-out put options, i.e. we then consider $S_0 \in \{40, 42.5, 45, 47.5, 49.5\}$ and $L = 50$. Finally, we fix times to maturity according to our previous scheme, i.e. we consider the maturities $\mathcal{T} \in \{0.25, 0.75, 1.5\}$. The results are summarized in Table 2.4 and Table 2.5.

Table 2.5: Theoretical up-and-out put values for $K = 45$, $\delta = 0.025$ and barrier level $L = 50$.

Up-and-Out Put Option Prices													
		Volatility Param. $\sigma = 0.2$						*Volatility Param.* $\sigma = 0.4$					
Parameters		*European*	*American*					*European*	*American*				
				N-th Order Approx.						*N-th Order Approx.*			
	S_0	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$
	40	4.981	5.105	5.089	5.104	5.106	5.105	6.039	6.096	6.084	6.092	6.098	6.097
(1)	42.5	3.055	3.110	3.100	3.108	3.110	3.110	4.319	4.355	4.347	4.351	4.355	4.355
$r = 4.88\%$	45	1.621	1.644	1.641	1.642	1.644	1.644	2.770	2.791	2.787	2.789	2.791	2.791
$\mathcal{T} = 0.25$	47.5	0.666	0.673	0.673	0.673	0.673	0.673	1.349	1.358	1.356	1.357	1.358	1.358
	49.5	0.122	0.123	0.123	0.123	0.123	0.123	0.267	0.268	0.268	0.268	0.268	0.268
	40	5.296	5.552	5.529	5.544	5.552	5.553	6.716	6.877	6.868	6.866	6.875	6.879
(2)	42.5	3.663	3.811	3.798	3.804	3.811	3.812	4.961	5.072	5.067	5.063	5.071	5.074
$r = 4.88\%$	45	2.272	2.351	2.346	2.347	2.352	2.352	3.265	3.335	3.332	3.329	3.334	3.336
$\mathcal{T} = 0.75$	47.5	1.073	1.107	1.105	1.105	1.107	1.108	1.615	1.649	1.648	1.646	1.649	1.650
	49.5	0.208	0.214	0.214	0.214	0.214	0.214	0.321	0.328	0.327	0.327	0.328	0.328
	40	5.396	5.856	5.842	5.845	5.853	5.857	6.789	7.131	7.142	7.122	7.126	7.130
(3)	42.5	3.860	4.152	4.146	4.142	4.149	4.154	5.040	5.285	5.294	5.277	5.280	5.284
$r = 4.88\%$	45	2.466	2.637	2.635	2.630	2.635	2.638	3.329	3.487	3.493	3.481	3.483	3.486
$\mathcal{T} = 1.50$	47.5	1.187	1.266	1.265	1.262	1.264	1.266	1.650	1.727	1.731	1.725	1.726	1.727
	49.5	0.231	0.246	0.246	0.246	0.246	0.246	0.328	0.343	0.344	0.343	0.343	0.343
RMSE ($\times 10^{-1}$)			–	**0.095**	**0.054**	**0.014**	**0.007**		–	**0.062**	**0.057**	**0.024**	**0.008**
CPU (sec.)			**1484.25**	**0.012**	**0.041**	**0.091**	**0.185**		**1483.96**	**0.012**	**0.041**	**0.101**	**0.204**

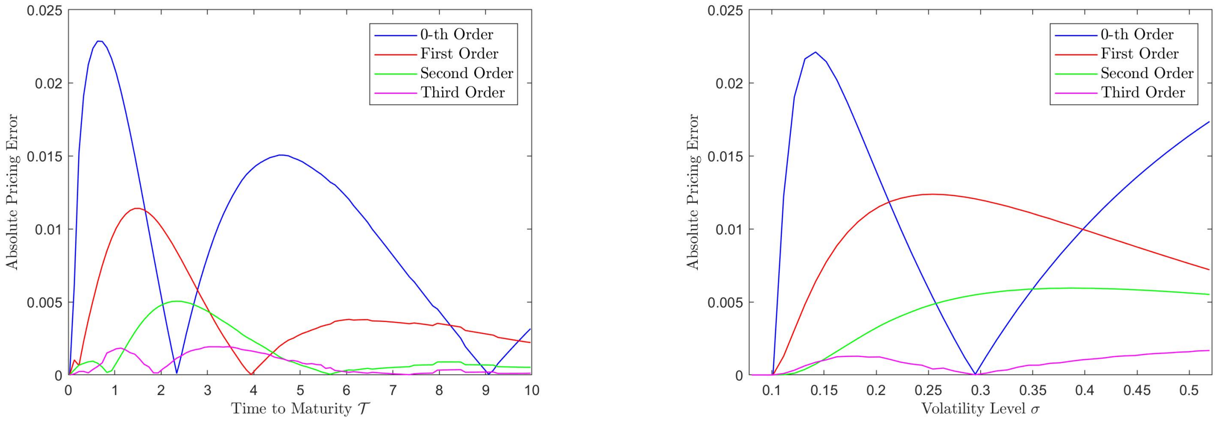
The simulation results show that our approximations for regular American barrier options have very similar properties to the ones obtained when analyzing our approximations for standard American options. As earlier, our higher order approximations outperform the 0-th order method in any cases where the early exercise premium does not become meaningless and increasing in theses cases the order of our approximations substantially reduces the pricing error made by our method. Additionally, we note that a high pricing accuracy can be obtained by relying on higher order approximations. All these findings are confirmed by Figure 2.3a and Figure 2.3b where we have plotted for $r = 0.0488$, $\delta = 0.025$, $S_0 = 40$, $K = 45$ and barrier level $L = 50$ the absolute up-and-out put pricing errors as functions of the time to maturity $\mathcal{T} \in (0, 10)$ and of the volatility level $\sigma \in (0.075, 0.525)$. Here, it is worth mentioning that our third order approximation exhibits a remarkable accuracy on the whole domains $\mathcal{T} \in (0, 10)$ and $\sigma \in (0.075, 0.525)$. Finally, we mention as earlier that increasing the order of our approximations leads to higher computational costs when executing the algorithm. However, we note that the costs of all our approximations – especially of our higher order approximations – is significantly lower than the costs of the respective versions for standard American options. This result is a direct consequence of the fact that, even for barrier options, European prices under the Black & Scholes model can be computed using simple formulae, while in Merton's model already standard European prices are expressed in terms of (infinite) series.

Table 2.6: Theoretical up-and-out put values for $K = 45$, $\delta = 0.025$ and barrier level $L = 50$.

Up-and-Out Put Option Prices								
Parameters		*European*	*American*					
					N-th Order Approx.			
	S_0	*Europ. Price*	*Bench-mark*	Mod. Quad. Approx.	$N = 0$	$N = 1$	$N = 2$	$N = 3$
(1)	40	4.981	5.105	5.090	5.089	5.104	5.106	5.105
$r = 4.88\%$	42.5	3.055	3.110	3.101	3.100	3.108	3.110	3.110
$\sigma = 0.2$	45	1.621	1.644	1.641	1.641	1.642	1.644	1.644
$\mathcal{T} = 0.25$	47.5	0.666	0.673	0.673	0.673	0.673	0.673	0.673
	49.5	0.122	0.123	0.123	0.123	0.123	0.123	0.123
(2)	40	5.296	5.552	5.537	5.529	5.544	5.552	5.553
$r = 4.88\%$	42.5	3.663	3.811	3.805	3.798	3.804	3.811	3.812
$r = 4.88\%$	45	2.272	2.351	2.351	2.346	2.347	2.352	2.352
$\mathcal{T} = 0.75$	47.5	1.073	1.107	1.108	1.105	1.105	1.107	1.108
	49.5	0.208	0.214	0.214	0.214	0.214	0.214	0.214
(3)	40	5.396	5.856	5.870	5.842	5.845	5.853	5.857
$r = 4.88\%$	42.5	3.860	4.152	4.169	4.146	4.142	4.149	4.154
$\sigma = 0.2$	45	2.466	2.637	2.651	2.635	2.630	2.635	2.638
$\mathcal{T} = 1.50$	47.5	1.187	1.266	1.274	1.265	1.262	1.264	1.266
	49.5	0.231	0.246	0.248	0.246	0.246	0.246	0.246
RMSE ($\times 10^{-1}$)			–	**0.093**	**0.095**	**0.054**	**0.014**	**0.007**
CPU (sec.)			**1484.25**	**0.012**	**0.012**	**0.041**	**0.091**	**0.185**

To additionally illustrate the quality of our algorithm, we next provide in Table 2.6 a comparison of numerical results between our approximations and comparable methods. Although the Barone-Adesi & Whaley extension of [AIL03] provides an important reference point for our approximations, we first note that it is already discussed throughout all our simulation studies since it corresponds to our 0-th order version for American barrier options. Therefore, we focus on a comparison of results obtained with the modified quadratic approximation of [CKKK07] and with our approximations of order up to three. Here, we rely once again on the parameter choices of [CKKK07], i.e. we take $\sigma = 0.2$, $r = 0.0488$, $\delta = 0.025$, $K = 45$, $L = 50$, and initial values $S_0 \in \{40, 42.5, 45, 47.5, 49.5\}$. Nevertheless, we note that considering other parameters does not substantially change the results. This is in line with the analysis presented in Figure 2.3.

The results in Table 2.6 show a clear dominance of all our higher order approximations over the modified quadratic scheme of [CKKK07]. In fact, while the latter method provides a marginal increase in accuracy compared to the Barone-Adesi & Whaley extension of [AIL03] (i.e. compared to our 0-th order version),



(a) Time to Maturity Graph: $\mathcal{T} \in (0, 10)$. **(b)** Volatility Graph: $\sigma \in (0.075, 0.525)$.

Figure 2.3: Absolute up-and-out put option pricing errors as functions of the time to maturity $\mathcal{T} \in (0, 10)$, and the volatility $\sigma \in (0.075, 0.525)$. The remaining parameters are chosen as: $\sigma = 0.2$, $r = 0.0488$, $\delta = 0.025$, $S_0 = 40$, $K = 45$, $L = 50$ and $\mathcal{T} = 0.75$.

our higher order approximations substantially decrease the pricing errors. This is clearly reflected in the resulting RMSEs. In terms of efficiency (CPU), the modified quadratic approximation of [CKKK07] has the advantage to be very much comparable to the Barone-Adesi & Whaley scheme. This is however not surprising, as this method essentially replicates the Barone-Adesi & Whaley ansatz of [AIL03] while including an additional parameter.

Table 2.7: Theoretical up-and-out put values for $K = 50$, $\delta = 0.06$ and barrier level $L = 49$.

Up-and-Out Put Option Prices													
		Volatility Param. $\sigma = 0.2$						*Volatility Param. $\sigma = 0.4$*					
Parameters		*European*	*American*					*European*	*American*				
				N-th Order Approx.						*N-th Order Approx.*			
	S_0	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$	*Europ. Price*	*Bench-mark*	$N = 0$	$N = 1$	$N = 2$	$N = 3$
	35	14.829	15.000	15.000	15.000	15.000	15.000	14.810	15.000	15.000	15.000	15.000	15.000
(1)	40	9.966	10.013	10.020	10.012	10.017	10.013	9.918	10.006	10.012	10.006	10.006	10.006
$r = 4.88\%$	45	5.046	5.055	5.062	5.054	5.058	5.055	4.985	5.017	5.022	5.018	5.017	5.017
$\mathcal{T} = 0.50$	48	2.025	2.027	2.029	2.027	2.028	2.027	2.000	2.008	2.009	2.008	2.008	2.008
	48.5	1.514	1.515	1.516	1.515	1.515	1.515	1.500	1.504	1.505	1.504	1.504	1.504
	35	14.647	15.000	15.000	15.000	15.000	15.000	14.575	15.000	15.000	15.000	15.000	15.000
(3)	40	9.889	10.020	10.036	10.020	10.028	10.020	9.775	10.006	10.014	10.006	10.006	10.006
$r = 4.88\%$	45	5.027	5.064	5.078	5.065	5.071	5.065	4.924	5.017	5.023	5.018	5.017	5.017
$\mathcal{T} = 1.00$	48	2.021	2.030	2.033	2.030	2.031	2.030	1.985	2.008	2.009	2.008	2.008	2.008
	48.5	1.512	1.516	1.518	1.516	1.517	1.516	1.493	1.504	1.505	1.504	1.504	1.504
	35	14.450	15.000	15.000	15.000	15.000	15.000	14.319	15.000	15.000	15.000	15.000	15.000
(4)	40	9.786	10.021	10.044	10.023	10.021	10.021	9.614	10.006	10.014	10.006	10.006	10.006
$r = 4.88\%$	45	4.987	5.067	5.085	5.068	5.066	5.066	4.852	5.017	5.023	5.018	5.017	5.017
$\mathcal{T} = 1.50$	48	2.011	2.030	2.035	2.031	2.030	2.030	1.967	2.008	2.009	2.008	2.008	2.008
	48.5	1.507	1.516	1.519	1.517	1.516	1.516	1.484	1.504	1.505	1.504	1.504	1.504
RMSE ($\times 10^{-2}$)			–	**0.978**	**0.056**	**0.304**	**0.021**		–	**0.428**	**0.031**	**0.003**	**0.002**
CPU (sec.)			**1493.02**	**0.012**	**0.039**	**0.083**	**0.167**		**1515.77**	**0.012**	**0.041**	**0.088**	**0.173**

We lastly turn to reverse up-and-out put options, i.e. we look at up-and-out put options in the case where the barrier level L and strike price K have the following relation: $L < K$. This situation is characterized by the fact that an American up-and-out put option holder will always exercise his option at the time the price process touches the barrier level, since this allows him a recovery of $K - L$. Hence, the American up-and-out put option turns in this case into an option with rebate as defined in Section 2.4.1.1 and dealing with this situation can be done accordingly.[15]

To allow for a better comparability of our results, we rely on the parameter choice made in [CKKK07], i.e. we take $\sigma \in \{0.2, 0.4\}$, $r = 0.0488$, $\delta = 0.06$, $K = 50$, $S_0 \in \{35, 40, 45, 48, 48.5\}$ and barrier level $L = 49$. Our approximations are implemented based on the ansatz offered in Section 2.4.1.1. In particular, this means that we first compute the price of the relevant European up-and-out put options with rebate $\mathcal{R}(K, L) := (K - L)^+$ and subsequently compute the corresponding early exercise premium via our approximations. Consequently, when referring to the European price we always think of rebate-type options. The results are summarized in Table 2.7.

As earlier, our simulation results show a clear dominance of the higher order approximations over the 0-th order algorithm. However, compared to the case of regular options, our higher order approximations seem to provide even more accuracy. This is easily deduced by comparing the RMSEs and noting that we have used different scaling parameters. Additionally, the results are consistent with the observations made so far for both standard American options as well as regular American barrier options: When using higher order approximations American-type options are priced with a high accuracy and increasing the order of our method generally leads to substantially more precision. Finally, we note that all these findings as well as the consistency obtained among the results suggest that applying the same method to other types of derivatives – for instance to lookback options, as done in [CKKK07] – is expected to deliver similar conclusions. However, since the main techniques would not differ much from the ones presented here, we do not detail these extensions.

2.6 Conclusion

The present article extended the current literature on pricing American-type options in two directions. First, we have considered the problem of pricing standard American options in jump-diffusion models. Here, we have extended the ansatz introduced under the Black & Scholes framework in [FMRZ15] to a model of constant jumps as well as to Merton's jump-diffusion model. The resulting approximations offer a generalization of the method proposed in [Ba91] and allow for a considerable increase in accuracy, when compared with the latter method. Secondly, we have considered the pricing of American barrier options under the model of Black & Scholes. Here, we have offered a generalization of the methods proposed in [AIL03] and [CKKK07] that is based on the techniques developed in the context of standard American options. We have tested all our approximations of up to order three using numerical simulations. Our numerical analysis showed a clear dominance of higher order approximations over their respective 0-th order version and revealed that significantly more pricing accuracy is obtained when relying on approximations of the first few orders. Additionally, they suggested that increasing the order of any approximation by one generally refines the pricing precision, however that this happens at the expense of greater computational complexity.

[15] We recall few central results for the pricing of options with rebates in Appendix C (cf. Section 2.7.3).

2.7 Appendices

2.7.1 Appendix A: Constant Jump Model

Let us review few well-known results on European call options that are crucially needed in the implementation of our N-th order approximations under Model (2.3.35). Being close to the Black & Scholes model, Model (2.3.35) is particularly manageable and many properties can be derived by slightly adapting their counterparts in the Black & Scholes framework. Using standard methods, one derives in particular that $\mathcal{C}_E(\mathcal{T}, x; K)$, the price of a European call option on $(S_t)_{t\geq 0}$ having maturity $\mathcal{T} \geq 0$, initial value $S_0 = x \geq 0$ and strike price $K \geq 0$, equals

$$\mathcal{C}_E(\mathcal{T}, x; K) = \sum_{n=0}^{\infty} e^{-(\lambda+r)\mathcal{T}} \frac{(\lambda\mathcal{T})^n}{n!} \mathcal{BS}\left(xe^{(r-\delta-\lambda(e^{\varphi}-1)+\frac{n\varphi}{\mathcal{T}})\mathcal{T}}, \sigma, \mathcal{T}; K\right), \tag{A.2.1}$$

where

$$\mathcal{BS}\left(X, \Sigma, T; K\right) := X\mathcal{N}\left(d_1\left(\frac{X}{K}, \Sigma, T\right)\right) - K\mathcal{N}\left(d_2\left(\frac{X}{K}, \Sigma, T\right)\right), \tag{A.2.2}$$

$\mathcal{N}(\cdot)$ denotes the standard normal CDF and

$$d_1\left(y, \varsigma, s\right) := \frac{1}{\sqrt{\varsigma^2 s}} \log(y) + \frac{1}{2}\sqrt{\varsigma^2 s}, \qquad d_2(y, \varsigma, s) := d_1(y, \varsigma, s) - \sqrt{\varsigma^2 s}. \tag{A.2.3}$$

For $\partial_x \mathcal{C}_E(\cdot)$, we first obtain from the Black & Scholes/Garman & Kohlhagen model (cf. [GK83]) that

$$\partial_x \left[e^{-r\mathcal{T}} \mathcal{BS}\left(xe^{(r-\delta-\lambda(e^{\varphi}-1)+\frac{n\varphi}{\mathcal{T}})\mathcal{T}}, \sigma, \mathcal{T}; K\right)\right] = e^{-(\delta+\lambda(e^{\varphi}-1)-\frac{n\varphi}{\mathcal{T}})\mathcal{T}}\, \mathcal{N}\left(d_1\left(\frac{xe^{(r-\delta-\lambda(e^{\varphi}-1)+\frac{n\varphi}{\mathcal{T}})\mathcal{T}}}{K}, \sigma, \mathcal{T}\right)\right)$$

and see that, for any $\mathcal{T} \in [0, T]$, we have

$$\left|\partial_x \left[e^{-r\mathcal{T}} \mathcal{BS}\left(xe^{(r-\delta-\lambda(e^{\varphi}-1)+\frac{n\varphi}{\mathcal{T}})\mathcal{T}}, \sigma, \mathcal{T}; K\right)\right]\right| \leq e^{-(\delta+\lambda(e^{\varphi}-1)-\frac{n\varphi}{\mathcal{T}})\mathcal{T}}.$$

The latter condition allows us to interchange differentiation and summation in the above series representation by means of the dominated convergence theorem and gives us finally that

$$\partial_x \mathcal{C}_E(\mathcal{T}, x; K) = \sum_{n=0}^{\infty} e^{-(\delta+\lambda e^{\varphi})\mathcal{T}}\, \frac{(\lambda\mathcal{T}e^{\varphi})^n}{n!}\, \mathcal{N}\left(d_1\left(\frac{xe^{(r-\delta-\lambda(e^{\varphi}-1)+\frac{n\varphi}{\mathcal{T}})\mathcal{T}}}{K}, \sigma, \mathcal{T}\right)\right). \tag{A.2.4}$$

Using the same approach, higher order Greeks can be also derived from the corresponding Black & Scholes properties. While both $\mathcal{C}_E(\cdot)$ and $\partial_x \mathcal{C}_E(\cdot)$ are explicitly needed in the derivation of our approximations, higher order Greeks can help improving the stability of the higher order algorithms (cf. Remark 2.1.).

2.7.2 Appendix B: Merton's Jump-Diffusion Model

Following the line of Appendix A (cf. Section 2.7.1), we now briefly recall central results on European options under Merton's jump-diffusion model (cf. [Me76]). First, one obtains that the price of a European call option under Merton's Model (2.3.47) having maturity $\mathcal{T} \geq 0$, initial value $S_0 = x \geq 0$ and strike price $K \geq 0$, equals

$$\mathcal{C}_E^{\mathcal{M}}(\mathcal{T}, x; K) = \sum_{n=0}^{\infty} e^{-(\lambda+r)\mathcal{T}} \frac{(\lambda\mathcal{T})^n}{n!} \mathcal{BS}\left(xe^{(r-\delta-\lambda\zeta+\frac{n\log(1+\zeta)}{\mathcal{T}})\mathcal{T}}, \Sigma_n, \mathcal{T}; K\right), \tag{A.2.5}$$

where $\Sigma_n := \sqrt{\sigma^2 + \frac{n\sigma_{\mathcal{M}}^2}{\mathcal{T}}}$ and we have used Notation (A.2.2) and (A.2.3). Secondly, one readily computes the delta $\partial_x \mathcal{C}_E^{\mathcal{M}}(\cdot)$ and obtain that it equals

$$\partial_x \mathcal{C}_E^{\mathcal{M}}(\mathcal{T}, x; K) = \sum_{n=0}^{\infty} e^{-(\delta+\lambda(1+\zeta))\mathcal{T}} \, \frac{\big(\lambda\mathcal{T}(1+\zeta)\big)^n}{n!} \, \mathcal{N}\left(d_1\left(\frac{xe^{(r-\delta-\lambda\zeta+\frac{n\log(1+\zeta)}{\mathcal{T}})\mathcal{T}}}{K}, \Sigma_n, \mathcal{T}\right)\right). \tag{A.2.6}$$

As in the model of constant jumps, we note that both $\mathcal{C}_E^{\mathcal{M}}(\cdot)$ and $\partial_x \mathcal{C}_E^{\mathcal{M}}(\cdot)$ are explicitly needed in the derivation of our approximations while further, higher order Greeks can be derived to help improving the stability of the higher order algorithms.

2.7.3 Appendix C: Barrier Options with Rebate

In this Appendix, we briefly review some well-known results to arrive at a valuation formula for

$$\mathcal{R}(K, L)\mathbb{E}_x^{\mathbb{Q}}\left[B_{\tau_L}(r)^{-1}\mathbb{1}_{\{\tau_L \le \mathcal{T}\}}\right], \tag{A.2.7}$$

the rebate term in (2.4.2). Further details can be found in the well-written book [JYC06].

First, we note that, for $S_0 > L$,

$$\begin{aligned} \tau_L &:= \inf\{t > 0 : \; S_t \le L\} && \text{(A.2.8)} \\ &= \inf\{t > 0 : \; \nu t + W_t \le y\}, && \text{(A.2.9)} \end{aligned}$$

where $(W_t)_{t\ge 0}$ is a Brownian motion and ν, y are given by $\nu := \frac{1}{\sigma}\big(r - \delta - \frac{1}{2}\sigma^2\big)$ and $y := \frac{1}{\sigma}\log\left(\frac{L}{S_0}\right) < 0$. Hence, computing the rebate term reduces to valuing a particular Laplace transform for the hitting time of a drifted Brownian motion. These results are known in closed form. Indeed, we have that, for $y < 0$,

$$\mathbb{E}_{S_0}^{\mathbb{Q}}\left[e^{-r\tau_L}\mathbb{1}_{\{\tau_L \le \mathcal{T}\}}\right] = e^{(\nu-\gamma)y}\mathcal{N}\left(\frac{-\gamma\mathcal{T} + y}{\sqrt{\mathcal{T}}}\right) + e^{(\nu+\gamma)y}\mathcal{N}\left(\frac{\gamma\mathcal{T} + y}{\sqrt{\mathcal{T}}}\right), \tag{A.2.10}$$

where γ is chosen to satisfy

$$\gamma = \pm\sqrt{2r + \nu^2}. \tag{A.2.11}$$

Therefore, Formula (A.2.10) provides us with a closed form expression for the term in (A.2.7) and similar results can be obtained in the case of an up-barrier (cf. [JYC06]).

As before, we note that these closed form results are crucially needed for the computation of European barrier-type options with rebate in the implementation of our N-th order approximations.

Acknowledgements: I would like to thank Walter Farkas and Giovanni Barone-Adesi for their advice as well as two anonymous reviewers and Christoph Reisinger for their constructive comments. I also thank Jérôme Detemple, Sander Willems, Alexander Smirnow, Urban Ulrych, Jakub Rojcek, Matthias Feiler and the participants of the Gerzensee Research Days 2018 for their valuable suggestions.

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Chapter 3

Valuing Tradeability in Exponential Lévy Models

A version of this paper is published as: *Mathys Ludovic, Valuing Tradeability in Exponential Lévy Models, Quantitative Finance and Economics* **4***(3), 459-488, 2020.*

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Abstract

The present article provides a novel theoretical way to evaluate tradeability in markets of ordinary exponential Lévy type. We consider non-tradeability as a particular type of market illiquidity and investigate its impact on the price of the assets. Starting from an adaption of the continuous-time optional asset replacement problem initiated by McDonald and Siegel (cf. [MS86]), we derive tradeability premiums and subsequently characterize them in terms of free-boundary problems. This provides a simple way to compute non-tradeability values, e.g. by means of standard numerical techniques, and, in particular, to express the price of a non-tradeable asset as a percentage of the price of a tradeable equivalent. Our approach is illustrated via numerical examples where we discuss various properties of the tradeability premiums.

Keywords: Tradeability, Liquidity, Exponential Lévy Processes, Real Options, Maturity-Randomization, Optimal Stopping, Free-Boundary Problems.

MSC (2010) Classification: 91B70, 91G20, 91G80.

JEL Classification: C32, G12, G13.

3.1 Introduction

Market liquidity and related risks have played an important role since the emergence of financial markets and their relevance for various types of financial activities has been noticed by academics since many years. Recently, the financial crisis has made it again clear how valuable and important market liquidity is. While trading costs rose for many assets dramatically, other assets could not be even traded for several months. Under such circumstances, liquidating an open position either became prohibitively expensive or was just impossible so that many investors were forced to sit on their positions and accumulated losses. In view of these incidents, it is not surprising that investors apprehend liquidity-related issues and usually demand a price discount when purchasing illiquid assets. This behavior is well documented by a vast body of empirical literature that started with the seminal articles of Amihud and Mendelson (cf. [AM86] and [AM89]).

In the literature, market liquidity usually either refers to the possibility to sell and buy – thus just to trade – financial assets on their respective markets or to the ability to trade them without initiating significant changes on the market. Although these two concepts are quite close to each other, there is an essential difference between them. While the first view merely understands liquidity in the sense of absolute tradeability, the second approach includes the effects that trading may trigger on the markets. For this reason, considering liquidity in the sense of the second approach generally offers more modeling flexibility than focussing on the first view. This could possibly explain why only little theoretical work analyzes the impact of non-tradeability on asset prices.[1] Indeed, despite the importance of non-tradeability issues, most theoretical models focus on the second view and capture (il-)liquidity by modeling the costs associated with trading the assets. Examples include the financial economics models[2] of Amihud and Mendelson (cf. [AM86]) and Acharya and Pedersen (cf. [AP05]) as well as many articles in the mathematical literature on liquidity, such as [Ja94], [CJP04] and [CR07] just to name a few.[3] In addition to the scarcity of the literature on tradeability, theoretical articles dealing with non-tradeability issues mostly derive premiums based on optimal selling strategies and could, therefore, only offer limited explanations for the existence and, in particular, the size of tradeability premiums. This includes the works of Longstaff (cf. [Lo95] and [Lo18]), as well as the articles of Koziol and Sauerbier (cf. [KS07]) and of Chesney and Kempf (cf. [CK12]). For these reasons, there is a clear need for alternative models that complement this literature and its current approaches. Such an alternative is proposed in the present article.

We propose a novel theoretical way to analyze the impact of non-tradeability on the price of assets in exponential Lévy markets. As we shall see, our framework starts from an adaption of the continuous-time optional asset replacement problem initiated in the seminal paper of McDonald and Siegel (cf. [MS86]). Considering an investor that holds an asset of (ordinary) exponential Lévy type and that faces the decision to replace it with an alternative investment project allows us to analyze two different tradeability scenarios for the asset: A fully liquid and a fully illiquid scenario. By assuming that the investor acts optimally in any of these scenarios, we derive absolute tradeability premiums as differences between the value of the replacement option in the respective scenarios and subsequently provide a free-boundary characterization of the latter premiums. This finally gives us a way to compute non-tradeability values, e.g. by means of standard numerical techniques, and, in particular, to express the price of an illiquid asset as a percentage of the price of a tradeable equivalent.

Our method has some similarities with the approaches taken in [Lo95] and [Lo18], [KS07] and [CK12]. As

[1] cf. [Lo95], [Lo18] and [CK12] for examples of articles tackling these issues and additional explanations on the challenges encountered when modeling non-tradeability.

[2] cf. [AMP05] for a survey of this literature.

[3] cf. [GRS11] for a survey of the mathematical literature on liquidity.

in these articles, valuing tradeability is linked to the opportunity costs of holding the asset. However, there are essential differences in the way the worthiness of tradeability is triggered. For instance, while the value of tradeability arises in [CK12] from the ability of traders to exploit temporary pricing inefficiencies in the market, tradeability enables one, in our model, to take advantage of the continuous possibility to invest in an alternative project. Therefore, instead of valuing tradeability merely out of optimal selling strategies, our approach considers reinvestment opportunities. In this sense, our model has a higher degree of completeness and provides more realistic bounds for the (individual) valuation of tradeability.

The remaining of this paper is structured as follows: In Section 3.2, we establish the general framework in which we model tradeability. This section essentially focuses on a proper introduction of the broad model as well as of the notation used in the rest of the paper. For this reason, the discussion therein does not include any tradeability aspects and the latter are only introduced in Sections 3.3 and 3.4. Sections 3.3 and 3.4 are both divided into two parts. While the first part introduces our tradeability modeling approaches, the second part deals with partial integro-differential equations (PIDEs) and ordinary integro-differential equations (OIDEs) for tradeability valuation. Here, our main results are Proposition 3.3 and Proposition 3.6 where free-boundary characterizations of the (absolute) tradeability premiums are provided. The importance of these propositions is illustrated in Section 3.5 where the respective free-boundary problems are solved for a particular model and numerical results are discussed. The paper concludes with Section 3.6. All proofs and complementary results are presented in the Appendices (Appendix A, B, C and D; Section 3.7).

3.2 General Framework and Notation

We start with a setting similar to that of the investment problem introduced in the seminal paper of McDonald and Siegel (cf. [MS86]): We consider the investment decision of an investor that holds an asset $(S_t)_{t\geq 0}$ and that has the option to replace it with an investment alternative. At any time $t \geq 0$, the investor can pay S_t to enter (or acquire a corresponding share of) an investment project that generates positive, net instantaneous cash-flow per unit of investment $(C_u)_{u\geq t}$ and has to make the decision to either continue holding the asset or to switch to the alternative project. As in [MS86], this asset replacement is understood as a continuous-time and irreversible decision to be taken. Whether or not the investment project is fully owned by the investor will not play any role in our analysis.[4]

3.2.1 Dynamics of the Initial Asset

We denote by r the risk-free interest rate, fix with $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ a filtered probability space – a chosen risk-neutral probability space[5] – and assume that the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Determining the properties of the asset replacement involves a complete description of its components, the initial asset and the alternative project. We start by characterizing the investor's initial investment: We assume that the investor's initial asset $(S_t)_{t\geq 0}$ trades on a usual market that is described, under the risk neutral measure $\mathbb{Q}$, by an (ordinary) exponential Lévy model, i.e. we assume that the price dynamics of the asset are given by

$$S_t = S_0 e^{X_t}, \qquad S_0 > 0,\, t \geq 0. \tag{3.2.1}$$

[4]We assume that the project's remuneration is proportional to the investment and, in particular, that the cash-flow generated out of the project does not depend on the type of ownership.

[5]It is well-known that exponential Lévy markets are incomplete as defined by Harrison and Pliska (cf. [HP81]). Specifying or discussing a particular choice of risk-neutral measure is not the sake of this article. Instead, we assume that a pricing measure under which our model has the required dynamics was previously fixed.

Here, the process $(X_t)_{t\geq 0}$ is an $\mathbf{F}$-Lévy process associated with a triplet (b_X, σ_X^2, Π_X), i.e. a càdlàg (right-continuous with left limits) process having independent and stationary increments and Lévy-exponent $\Psi_X(\cdot)$ defined, for $\theta \in \mathbb{R}$, by

$$\Psi_X(\theta) := -\log\left(\mathbb{E}^{\mathbb{Q}}\left[e^{i\theta X_1}\right]\right) = -ib_X\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\leq 1\}})\Pi_X(dy), \tag{3.2.2}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ refers to expectation with respect to the measure $\mathbb{Q}$. Applying the well-known Lévy-Itô decomposition theorem (cf. [Sa99], [Ap09]) allows one to separate $(X_t)_{t\geq 0}$ into its diffusion and jump parts: Indeed, there exists an $\mathbf{F}$-Brownian motion $(W_t^X)_{t\geq 0}$ and an independent Poisson random measure N_X on $[0,\infty) \times \mathbb{R} \setminus \{0\}$ having intensity measure Π_X, such that

$$X_t = b_X t + \sigma_X W_t^X + \int_{\mathbb{R}} y \, \bar{N}_X(t, dy), \qquad t \geq 0, \tag{3.2.3}$$

where we use for $t \geq 0$ and any Borel set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ the notation

$$\begin{aligned} N_X(t, A) &:= N_X((0,t] \times A), \\ \tilde{N}_X(dt, dy) &:= N_X(dt, dy) - \Pi_X(dy)dt, \\ \bar{N}_X(dt, dy) &:= \begin{cases} \tilde{N}_X(dt, dy), & \text{if } |y| \leq 1, \\ N_X(dt, dy), & \text{if } |y| > 1. \end{cases} \end{aligned}$$

This directly gives a corresponding factorization of the price dynamics $(S_t)_{t\geq 0}$ into exponentials of the diffusion and jump parts of $(X_t)_{t\geq 0}$.

Additionally, the Laplace exponent of the Lévy process $(X_t)_{t\geq 0}$ is defined, for any $\theta \in \mathbb{R}$ satisfying the condition $\mathbb{E}^{\mathbb{Q}}\left[e^{\theta X_1}\right] < \infty$, by the following identity:

$$\Phi_X(\theta) := \log\left(\mathbb{E}^{\mathbb{Q}}\left[e^{\theta X_1}\right]\right) = b_X\theta + \frac{1}{2}\sigma_X^2\theta^2 - \int_{\mathbb{R}} (1 - e^{\theta y} + \theta y \mathbb{1}_{\{|y|\leq 1\}})\Pi_X(dy). \tag{3.2.4}$$

In the sequel, we will assume that this quantity is at least for $\theta = 1$ well defined, i.e. that $\mathbb{E}^{\mathbb{Q}}\left[e^{X_1}\right] < \infty$, and finally require that $\Phi_X(1) \leq r$. The latter condition has an important feature: It is well-known that discounted, exponential Lévy models of the form of (3.2.1) have the martingale property if and only if the usual integrability condition[6] and additionally $\Phi_X(1) = r$ are satisfied (cf. [JYC06], [Ap09]). Hence, requiring $\Phi_X(1) \leq r$ to hold under the measure $\mathbb{Q}$ allows the asset to pay a (continuous) dividend and the discounted asset dynamics have the martingale structure only under a lower, adjusted discount factor $r - \tilde{r}$. Such dynamics are typically found in foreign exchange markets, where $\tilde{r}$ represents the foreign risk-free interest rate (cf. [GK83], [JC04]).

3.2.2 Dynamics of the Investment Alternative

We next turn to the investor's investment alternative. As we shall see in a moment, characterizing the investor's investment project reduces to specifying the dynamics of the process $(C_t)_{t\geq 0}$, the net instantaneous cash-flow generated out of a one-unit investment in the project. Indeed, once this process is specified the

[6] $\mathbb{E}^{\mathbb{Q}}\left[e^{X_t}\right] < \infty$ or, equivalently, $\int_{\{|y|>1\}} e^y \, \Pi_X(dy) < \infty$ (cf. [Sa99], Theorem 25.3). This is clearly satisfied by our assumptions.

project's value can be easily recovered by computing the expected net present value of the project's future cash-flows. Therefore, we start by determining the dynamics of the cash-flow process and assume that $(C_t)_{t\geq 0}$ follows under $\mathbb{Q}$ another exponential Lévy model of the form

$$C_t = C_0 e^{Y_t}, \qquad C_0 > 0,\, t \geq 0, \tag{3.2.5}$$

where $(Y_t)_{t\geq 0}$ denotes an $\mathbf{F}$-Lévy process with Lévy triplet (b_Y, σ_Y^2, Π_Y). As for $(X_t)_{t\geq 0}$, one obtains (by means of the Lévy-Itô decomposition theorem) a separation of $(Y_t)_{t\geq 0}$ into its diffusion and jump parts of the form

$$Y_t = b_Y t + \sigma_Y W_t^Y + \int_{\mathbb{R}} y \, \bar{N}_Y(t, dy), \qquad t \geq 0, \tag{3.2.6}$$

where $(W_t^Y)_{t\geq 0}$ denotes an $\mathbf{F}$-Brownian motion and N_Y a corresponding Poisson random measure on $[0,\infty)\times \mathbb{R}\setminus\{0\}$ that is independent of $(W_t^Y)_{t\geq 0}$. The dependence structure between the two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ (and so between both exponential Lévy models $(S_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$) is additionally fixed by assuming that the Poisson random measures N_X and N_Y are independent and that the Brownian parts $(W_t^X)_{t\geq 0}$ and $(W_t^Y)_{t\geq 0}$ have correlation coefficient $|\rho| \leq 1$, i.e. that $\left[W^X, W^Y\right]_t = \rho t$. As earlier, we require the existence of the Laplace exponent $\Phi_Y(1)$ and demand that $\Phi_Y(1) < r$.

3.2.3 Asset Replacement Dynamics

To finally derive the time-t value of the asset replacement, we first compute for any $t \geq 0$ the expected net present value of the future cash-flow generated out of a one-unit investment in the project, E_t: Using Fubini's theorem for conditional expectation and the dynamics (3.2.5), one obtains that

$$\begin{aligned} E_t &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-r(u-t)} C_u \, du \middle| \mathcal{F}_t\right] = \int_t^\infty e^{-r(u-t)}\, \mathbb{E}^{\mathbb{Q}}\left[C_u | \mathcal{F}_t\right] du \\ &= \int_t^\infty e^{-r(u-t)} C_t e^{(u-t)\Phi_Y(1)}\, du = C_t \int_t^\infty e^{-(r-\Phi_Y(1))(u-t)}\, du = \frac{C_t}{r-\Phi_Y(1)}. \end{aligned} \tag{3.2.7}$$

Hence, the dynamics of $(E_t)_{t\geq 0}$ are proportional to those of $(C_t)_{t\geq 0}$ and E_t equals, at any $t \geq 0$,

$$E_t = E_0 e^{Y_t}, \qquad E_0 = \frac{C_0}{r - \Phi_Y(1)}.$$

The time-t value of a one-unit investment in the project, V_t, is now easily deduced. Clearly, this value corresponds to the difference of the expected net present value of the future cash-flows generated out of a one-unit investment in the project, E_t, and 1, the costs of such an investment. As a consequence, we obtain by (3.2.7) that

$$V_t = E_t - 1 = \frac{C_t}{r - \Phi_Y(1)} - 1. \tag{3.2.8}$$

At any possible switching date $t \geq 0$, the investor holds the option to sell his asset and to reinvest its full proceeds in the alternative project. Hence, the investor's possible time-t level of investment corresponds to the value S_t of the asset currently held. This finally gives that the time-t value of the asset replacement, V_t^S, equals

$$V_t^S = S_t \cdot V_t = S_t\left(E_t - 1\right). \tag{3.2.9}$$

Remark 3.1.

i) Equation (3.2.9) describes a version of the asset replacement that is scaled to one unit of the initial asset. However, looking at more general holdings does not change the replacement problem significantly and any such problem can be easily reduced to the one-unit situation.

ii) Notice that we did not make any assumption on the exclusiveness of the investment project: The project may represent an investment opportunity that is linked to the investor – if one thinks of the investor as a company, this could represent for instance a company's internal project – and so that is unique and not necessarily available (at least not in the exact same conditions) to any other competitor. But also more standard and open investment alternatives could be considered. In this context, any evaluation of the investment alternative under the risk-neutral measure $\mathbb{Q}$ does not correspond to a real pricing attempt but merely serves as an assessment of the project from the point of view of a "typical investor" within the market.

♦

3.3 Valuing Tradeability: Deterministic Illiquidity Horizon

3.3.1 Generalities

Up to this point, our general framework did not include any element that aimed to model differences in tradeability. This should be addressed next. To this end, we fix a (deterministic) time horizon $T_D > 0$ and consider variants of the optional asset replacement problem introduced in Section 3.2 on the time interval $[0, T_D]$.[7] We assume that the investment project is available at any date $t \in [0, T_D]$ and derive tradeability premiums by varying the marketability of the initial asset $(S_t)_{t\geq 0}$ on $[0, T_D]$ and analyzing the behavior of an investor that acts optimally in the resulting asset replacement problem. Hereby we compare two scenarios:

1. An illiquid scenario, where any attempt to sell the asset $(S_t)_{t\geq 0}$ at time $t \in [0, T_D)$ fails and the investor has to make a new decision at T_D. Hence, $\mathcal{T} := T_D - t$ is interpreted as illiquidity horizon.
2. A liquid scenario, where the tradeability of the investor's asset is guaranteed at any date $t \in [0, T_D]$.

Remark 3.2.

It is important to note that the present tradeability valuation approach is in line with [Lo95], [Lo18], and [CK12], and therefore understands tradeability to only occur at very few points in time. Under this assumption, restricting the analysis to the first illiquidity interval $[0, T_D]$ already provides sensible results while keeping a certain degree of tractability. Nevertheless, we emphasize that other approaches could be considered. As an example, analyzing a situation where non-tradeability is a temporary state beyond which the asset remains fully tradeable could be addressed as part of future research.

♦

3.3.1.1 Illiquid Scenario

We start by analyzing the investor's trading behavior in the illiquid scenario. Being modeled by ordinary exponential Lévy models, the processes $(S_t)_{t\geq 0}$ and $(E_t)_{t\geq 0}$ are assumed to be efficient. Hence, the investor

[7] Although $T_D = \infty$ could also be considered, it is not very meaningful. Therefore, we implicitly understand T_D to be finite and consider finite analogues of the optional asset replacement problem introduced in Section 3.2.

cannot anticipate future fluctuations and base his decision at any time $t \in [0, T_D]$ on his current information $\mathcal{F}_t$. At any time $t \in [0, T_D]$ at which $V_t^S > 0$, the investment project is more valuable than the asset and switching from S_t to $S_t E_t$, i.e. investing S_t in the project, provides an immediate increase in wealth in the amount of $V_t^S > 0$. Since the investor can only switch, in the illiquid scenario, at $t = T_D$, he will do so if and only if $V_{T_D}^S > 0$. As a consequence, the time-t value of this switching option $\mathfrak{C}_{\mathbf{E}}(\cdot)$ is obtained as

$$\mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t) := \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\left[e^{-r\mathcal{T}}\left(V_{\mathcal{T}}^S \vee 0\right)\right] = \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\left[e^{-r\mathcal{T}} S_{\mathcal{T}}\left(E_{\mathcal{T}} - 1\right)^+\right], \tag{3.3.1}$$

where we denote by $\mathbb{E}^{\mathbb{Q}}_{s_0,e_0}[\cdot]$ the expectation under $\mathbb{Q}_{s_0,e_0}$, the probability measure under which $(S_t)_{t\geq 0}$ and $(E_t)_{t\geq 0}$ start at $S_0 = s_0$ and $E_0 = e_0$, respectively. This corresponds to the time-t value of a European exchange option.

3.3.1.2 Liquid Scenario

Deriving the investor's trading behavior in the liquid scenario can be done by the very same arguments. However, since the initial asset is now perfectly tradeable there are no restrictions on the investor's switching possibilities. Hence, the investor will choose a switching rule that maximizes his immediate increase in wealth in expectation. As a consequence, evaluating the switching option in the liquid scenario reduces to valuing an American exchange option $\mathfrak{C}_{\mathbf{A}}(\cdot)$ of the form

$$\mathfrak{C}_{\mathbf{A}}(\mathcal{T}, S_t, E_t) := \sup_{\tau\in\mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\left[e^{-r\tau}\left(V_{\tau}^S \vee 0\right)\right] = \sup_{\tau\in\mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\left[e^{-r\tau} S_{\tau}\left(E_{\tau} - 1\right)^+\right], \tag{3.3.2}$$

where $\mathfrak{T}_{[0,\mathcal{T}]}$ denotes the set of stopping times that take values in the time interval $[0, \mathcal{T}]$.

3.3.1.3 Tradeability Premium and Transformation

The above optimal trading strategies can now be used to value tradeability: Both options $\mathfrak{C}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}_{\mathbf{A}}(\cdot)$ yield a monetization of the benefits that can be generated out of the exchange opportunity within the respective tradeability scenarios. Since the asset's tradeability is the only changing parameter, any inequality in these benefits must be a consequence of its variation. Therefore, we identify the (absolute) time-t tradeability/liquidity[8] premium $\mathfrak{L}(\cdot)$ with the difference of $\mathfrak{C}_{\mathbf{A}}(\cdot)$ and $\mathfrak{C}_{\mathbf{E}}(\cdot)$, i.e. we set

$$\mathfrak{L}(\mathcal{T}, S_t, E_t) := \mathfrak{C}_{\mathbf{A}}(\mathcal{T}, S_t, E_t) - \mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t). \tag{3.3.3}$$

At this point, we already notice a few properties of the tradeability premium (3.3.3). First, it is clear that our tradeability premium substantially depends on the dynamics of the alternative project. Since the dynamics and characteristics of available projects depend themselves on the investor's relations, resources, etc., our tradeability premium results in an individual value.[9] In addition, this value provides a theoretical lower bound for the (individual) valuation of tradeability. Indeed, our setting examines investment alternatives that are irreversible, at least during the time horizon $[0, T_D]$ considered. However, reversible investment possibilities clearly exist in practice. Therefore, extending the analysis to investment projects that can be themselves exchanged against others would provide more accuracy in our valuation approach. This extension is left out and could be part of future research.

[8] As emphasized in the introduction, we understand liquidity in the sense of absolute tradeability and will use, from now on, both terms interchangeably.

[9] Remember that we evaluate the investment alternative under the risk-neutral measure. Therefore, the resulting tradeability premium provides an individual, though market-weighted value.

Remark 3.3.

Instead of considering absolute values, it is often more informative to look at relative quantities. For this reason, our numerical results in Section 3.5 will focus on figures related to the relative time-t tradeability premium, defined as

$$\mathfrak{L}_{Rel.}(\mathcal{T}, S_t, E_t) := \frac{\mathfrak{L}(\mathcal{T}, S_t, E_t)}{\mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t)} = \frac{\mathfrak{C}_{\mathbf{A}}(\mathcal{T}, S_t, E_t)}{\mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t)} - 1.$$

♦

Valuing both switching options $\mathfrak{C}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}_{\mathbf{A}}(\cdot)$ and so the tradeability premium $\mathfrak{L}(\cdot)$ under $\mathbb{Q}$, i.e. from the point of view of a "typical investor" in the market, reduces to the usual pricing procedure: First, we introduce, for any stopping time $\tau \in \mathfrak{T}_{[0,\infty)} \cup \{\infty\}$,[10] the following notations

$$\mathfrak{C}(\tau, S_t, E_t) := \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\left[e^{-r\tau}\left(V^S_\tau \vee 0\right)\right], \tag{3.3.4}$$

$$\mathfrak{C}^\star(\tau, E_t) := \mathfrak{C}(\tau, 1, E_t), \tag{3.3.5}$$

and show in Appendix A (cf. Section 3.7.1) that, under the new measure $\mathbb{Q}^{(1)}$ defined by the (1-)Esscher transform[11]

$$\left.\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} := \frac{e^{1\cdot X_t}}{\mathbb{E}^{\mathbb{Q}}\left[e^{1\cdot X_t}\right]} = e^{X_t - t\Phi_X(1)}, \tag{3.3.6}$$

the process $(Y_t)_{t\in[0,T]}$ is, for any finite time horizon $T > 0$, again a Lévy process with Lévy-exponent $\Psi^{(1)}_Y(\cdot)$ having the form

$$\Psi^{(1)}_Y(\theta) = -i(b_Y + \rho\sigma_X\sigma_Y)\theta + \frac{1}{2}\sigma_Y^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\le 1\}})\Pi_Y(dy). \tag{3.3.7}$$

Then, rewriting $\mathfrak{C}(\cdot)$ under the change of measure (3.3.6) while bearing in mind the dynamics (3.2.1), (3.2.9) and (3.3.6) readily provides, for any $T > 0$, the expression

$$\mathfrak{C}(T\wedge\tau, S_t, E_t) = S_t \cdot \mathfrak{C}^\star(T\wedge\tau, E_t) = S_t \cdot \mathbb{E}^{\mathbb{Q}^{(1)}}_{E_t}\left[e^{-(r-\Phi_X(1))(T\wedge\tau)}\left(E_{T\wedge\tau} - 1\right)^+\right], \tag{3.3.8}$$

where $\mathbb{E}^{\mathbb{Q}^{(1)}}_{e_0}[\cdot]$ denotes expectation under $\mathbb{Q}^{(1)}_{e_0}$, the probability measure (associated to $\mathbb{Q}^{(1)}$ and) under which $(E_t)_{t\ge 0}$ starts at $E_0 = e_0$. This latter equation substantially simplifies the valuation problem for both $\mathfrak{C}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}_{\mathbf{A}}(\cdot)$. Indeed, combining the relations

$$\mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t) = \mathfrak{C}(\mathcal{T}, S_t, E_t) \quad \text{and} \quad \mathfrak{C}_{\mathbf{A}}(\mathcal{T}, S_t, E_t) = \sup_{\tau\in\mathfrak{T}_{[0,\mathcal{T}]}} \mathfrak{C}(\tau, S_t, E_t) \tag{3.3.9}$$

with (3.3.8) while introducing the notations $\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T}, E_t) := \mathfrak{C}_{\mathbf{E}}(\mathcal{T}, 1, E_t)$ and $\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, E_t) := \mathfrak{C}_{\mathbf{A}}(\mathcal{T}, 1, E_t)$ allows us to rewrite

$$\mathfrak{C}_{\mathbf{E}}(\mathcal{T}, S_t, E_t) = S_t \cdot \mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T}, E_t) = S_t \cdot \mathbb{E}^{\mathbb{Q}^{(1)}}_{E_t}\left[e^{-(r-\Phi_X(1))\mathcal{T}}\left(E_{\mathcal{T}} - 1\right)^+\right], \tag{3.3.10}$$

$$\mathfrak{C}_{\mathbf{A}}(\mathcal{T}, S_t, E_t) = S_t \cdot \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, E_t) = S_t \cdot \sup_{\tau\in\mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}^{(1)}}_{E_t}\left[e^{-(r-\Phi_X(1))\tau}\left(E_{\tau} - 1\right)^+\right]. \tag{3.3.11}$$

[10] At $t = \infty$ we set $S_t := E_t := 0$. This is just for the sake of accuracy as it will not play a real role in this article.

[11] The Esscher transform was first introduced 1932 by Esscher and later established in the theory of option pricing by Gerber and Shiu (cf. [GS94]). An economic interpretation of this pricing technique in the continuous-time framework can be found in [GS94].

Hence, both switching options are linear in the value S_t of the asset initially held. Furthermore, the scaled versions $\mathfrak{C}^{\star}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^{\star}_{\mathbf{A}}(\cdot)$ correspond to simple European and American-type options written on the exponential Lévy process $(E_t)_{t\in[0,T_D]}$. Consequently, valuing – under $\mathbb{Q}$ – any of the switching options $\mathfrak{C}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}_{\mathbf{A}}(\cdot)$ reduces – under $\mathbb{Q}^{(1)}$ – to the consideration of corresponding valuation problems for simple options on the exponential Lévy model

$$E_t = E_0 e^{Y_t}, \qquad E_0 > 0,\, t \in [0, T_D],$$

with Lévy exponent $\Psi^{(1)}_Y(\cdot)$ defined as in (3.3.7), risk-free interest rate $\tilde{r} := r - \Phi_X(1)$, and strike price $K := 1$.

3.3.2 PIDEs for Tradeability Valuation

Our next goal consists in deriving partial integro-differential equations that can be used to value tradeability. As argued in the previous section, we focus from now on on the respective valuation problems for $\mathfrak{C}^{\star}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^{\star}_{\mathbf{A}}(\cdot)$. We then define

$$\mathfrak{L}^{\star}(\mathcal{T}, E_t) := \mathfrak{L}(\mathcal{T}, 1, E_t) = \mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, E_t) - \mathfrak{C}^{\star}_{\mathbf{E}}(\mathcal{T}, E_t) \tag{3.3.12}$$

and recover $\mathfrak{L}(\cdot)$ from its scaled version $\mathfrak{L}^{\star}(\cdot)$ by means of the obvious relation

$$\mathfrak{L}(\mathcal{T}, S_t, E_t) = S_t \cdot \mathfrak{L}^{\star}(\mathcal{T}, E_t). \tag{3.3.13}$$

Remark 3.4.

Note that we can also express the relative time-t tradeability premium, using the above notation, as

$$\mathfrak{L}_{Rel.}(\mathcal{T}, S_t, E_t) = \mathfrak{L}_{Rel.}(\mathcal{T}, 1, E_t) = \frac{\mathfrak{L}^{\star}(\mathcal{T}, E_t)}{\mathfrak{C}^{\star}_{\mathbf{E}}(\mathcal{T}, E_t)} = \frac{\mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, E_t)}{\mathfrak{C}^{\star}_{\mathbf{E}}(\mathcal{T}, E_t)} - 1.$$

♦

In what follows, we will always assume that the second moment of the ($\mathbb{Q}^{(1)}$-)Lévy model $(E_t)_{t\in[0,T_D]}$ exists, or equivalently (cf. [Sa99], Theorem 25.3) that

$$\int_{\{|y|>1\}} e^{2y}\, \Pi_Y(dy) < \infty, \tag{3.3.14}$$

and note that this is a weak assumption that could be even relaxed (cf. [CV05a]). We start by determining the dynamics of the process $(E_t)_{t\in[0,T_D]}$ under the measure $\mathbb{Q}^{(1)}$. This is done using Itô's Lemma and readily gives that

$$dE_t = E_{t-}\left(\Phi^{(1)}_Y(1)dt + \sigma_Y d\tilde{W}^Y_t + \int_{\mathbb{R}} (e^y - 1)\tilde{N}_Y(dt, dy)\right), \tag{3.3.15}$$

where $(\tilde{W}^Y_t)_{t\in[0,T_D]}$ denotes a $\mathbb{Q}^{(1)}$-Brownian motion (cf. Appendix A; Section 3.7.1) and $\Phi^{(1)}_Y(\cdot)$ refers to the Laplace-exponent of $(Y_t)_{t\in[0,T_D]}$ under $\mathbb{Q}^{(1)}$.[12] Therefore, whenever well-defined, its infinitesimal generator

[12] Note that the existence of $\Phi^{(1)}_Y(1)$ directly follows from our initial assumptions, since the measure change defined by (3.3.6) does not alter the jump component of $(Y_t)_{t\geq 0}$ and we initially assumed that $\int_{\{|y|>1\}} e^y\, \Pi_Y(dy) < \infty$.

is a partial integro-differential operator obtained, for $V : [0, T_D] \times \mathbb{R} \to \mathbb{R}$, by

$$\begin{aligned}\mathcal{A}_E V(\mathcal{T}, x) &:= \lim_{t \downarrow 0} \frac{\mathbb{E}_x^{\mathbb{Q}^{(1)}}\big[V(\mathcal{T}, E_t)\big] - V(\mathcal{T}, x)}{t} \\ &= \frac{1}{2}\sigma_Y^2 x^2 \partial_x^2 V(\mathcal{T}, x) + \Phi_Y^{(1)}(1) x \partial_x V(\mathcal{T}, x) \\ &\qquad + \int_{\mathbb{R}} \big[V(\mathcal{T}, xe^y) - V(\mathcal{T}, x) - x(e^y - 1)\partial_x V(\mathcal{T}, x)\big]\Pi_Y(dy).\end{aligned} \tag{3.3.16}$$

3.3.2.1 PIDE I: Illiquid Scenario

We first deal with the illiquid scenario and rewrite, for $(\mathcal{T}, x) \in [0, T_D] \times [0, \infty)$, the European-type switching option in the form

$$\mathfrak{C}_{\mathbf{E}}^{\star}(\mathcal{T}, x) = \mathbb{E}_x^{\mathbb{Q}^{(1)}}\big[\, \big(\bar{E}_{\mathcal{T}} - 1\big)^+ \,\big], \tag{3.3.17}$$

where $(\bar{E}_t)_{t \in [0, T_D]}$ refers to the (strong) Markov process[13] obtained by "killing" the sample path of $(E_t)_{t \in [0, T_D]}$ at the proportional rate $\tilde{r} := r - \Phi_X(1)$. The process' transition probabilities are then given by

$$\mathbb{Q}_x^{(1)}\big(\bar{E}_t \in A\big) = \mathbb{E}_x^{\mathbb{Q}^{(1)}}\big[e^{-\tilde{r}t}\, \mathbb{1}_A(E_t)\big] \tag{3.3.18}$$

and we identify its cemetery state, without loss of generality, with $\partial \equiv 0$. Therefore, for any initial value $z = (\mathbf{t}, x) \in [0, T_D] \times [0, \infty)$, the process $(Z_t)_{t \in [0, \mathbf{t}]}$ defined via $Z_t := (\mathbf{t} - t, \bar{E}_t)$, $\bar{E}_0 = x$, is a strong Markov process with state domain given by $\mathcal{D}_{\mathbf{t}} := [0, \mathbf{t}] \times [0, \infty)$. Additionally, $\mathfrak{C}_{\mathbf{E}}^{\star}(\cdot)$ can be re-expressed as

$$\mathfrak{C}_{\mathbf{E}}^{\star}(\mathcal{T}, x) = V_E\big((\mathcal{T}, x)\big), \tag{3.3.19}$$

where the value function $V_E(\cdot)$ has the following representation under the measure $\mathbb{Q}_z^{(1),Z}$ having initial distribution $Z_0 = z$:

$$V_E(z) := \mathbb{E}_z^{\mathbb{Q}^{(1),Z}}\big[G(Z_{\tau_{\mathcal{S}}})\big], \qquad G(z) := (x - 1)^+, \tag{3.3.20}$$

and $\tau_{\mathcal{S}} := \inf\{t \geq 0 : Z_t \in \mathcal{S}\}$, $\mathcal{S} := \big(\{0\} \times [0, \infty)\big) \cup \big([0, \mathbf{t}] \times \{0\}\big)$, is a stopping time that satisfies $\tau_{\mathcal{S}} \leq \mathbf{t}$, under $\mathbb{Q}_z^{(1),Z}$ with $z = (\mathbf{t}, x)$. Furthermore, the stopping region $\mathcal{S}$ is for any $\mathbf{t} \in [0, T_D]$ a closed set in $\mathcal{D}_{\mathbf{t}}$. Therefore, standard arguments based on the strong Markov property of $(Z_t)_{t \in [0, \mathbf{t}]}$ (cf. [PS06]) imply that $V_E(\cdot)$ satisfies the following problem

$$\mathcal{A}_Z V_E(z) = 0, \qquad \text{on } \mathcal{D}_{T_D} \setminus \mathcal{S}, \tag{3.3.21}$$

$$V_E(z) = G(z), \qquad \text{on } \mathcal{S}, \tag{3.3.22}$$

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t \in [0, \mathbf{t}]}$. Additionally, we note that (for any suitable function $V : \mathcal{D}_{\mathbf{t}} \to \mathbb{R}$) the infinitesimal generator $\mathcal{A}_Z$ can be re-expressed as

$$\begin{aligned}\mathcal{A}_Z V\big((\mathbf{t}, x)\big) &= -\partial_{\mathbf{t}} V\big((\mathbf{t}, x)\big) + \mathcal{A}_{\bar{E}} V\big((\mathbf{t}, x)\big) \\ &= -\partial_{\mathbf{t}} V\big((\mathbf{t}, x)\big) + \mathcal{A}_E V\big((\mathbf{t}, x)\big) - \tilde{r} V\big((\mathbf{t}, x)\big).\end{aligned} \tag{3.3.23}$$

[13] It is well-known (cf. [PS06]) that the process $(\bar{E}_t)_{t \in [0, T_D]}$ defined this way preserves the (strong) Markov property of the underlying process $(E_t)_{t \in [0, T_D]}$.

Consequently, recovering $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$ via (3.3.19) finally gives the following PIDE:

$$-\partial_{\mathcal{T}}\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)+\mathcal{A}_E\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)-\tilde{r}\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)=0, \qquad \text{on } (0,T_D]\times(0,\infty), \tag{3.3.24}$$

$$\mathfrak{C}^\star_{\mathbf{E}}(0,x)=(x-1)^+, \qquad x\in[0,\infty). \tag{3.3.25}$$

Under few additional assumptions,[14] smoothness of the European-type switching option can be additionally shown. This is the content of the following proposition.

Proposition 3.1. *Assume that*

$$\text{either} \qquad \sigma_Y\neq 0 \qquad \text{or} \qquad \exists\alpha\in(0,2): \quad \liminf_{\epsilon\downarrow 0}\frac{1}{\epsilon^{2-\alpha}}\int_{-\epsilon}^{\epsilon}|y|^2\,\Pi_Y(dy)>0. \tag{3.3.26}$$

Then, the value of the European-type switching option under deterministic illiquidity horizon, $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$, is continuous on $[0,T_D]\times[0,\infty)$, $C^{1,2}$ on $(0,T_D)\times(0,\infty)$ and solves the partial integro-differential equation

$$-\partial_{\mathcal{T}}\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)+\mathcal{A}_E\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)-\tilde{r}\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)=0 \tag{3.3.27}$$

on $(0,T_D]\times(0,\infty)$ with initial condition

$$\mathfrak{C}^\star_{\mathbf{E}}(0,x)=(x-1)^+, \qquad x\in[0,\infty). \tag{3.3.28}$$

The proof of Proposition 3.1 is similar to that of Proposition 2 in [CV05a]. In this article, the authors work with exponential Lévy processes that have the martingale property. However, since $(E_t)_{t\in[0,T_D]}$ does not necessarily satisfy this property, we provide in Appendix B (cf. Section 3.7.2) an adaption of their proof that works in our more general context. Parts of the proof that do not involve any martingale argument will be directly referred to [CV05a].

3.3.2.2 PIDE II: Liquid Scenario

We next turn to the liquid scenario. As in the illiquid scenario, we derive a characterization of the American-type switching option $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$ by adapting well-established results for standard American options on exponential Lévy models. This leads to the next proposition.

Proposition 3.2. *The value of the American-type switching option under deterministic illiquidity horizon, $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$, is continuous on $[0,T_D]\times[0,\infty)$ and solves the non-linear Hamilton-Jacobi-Bellman (HJB) equation*

$$\max\Big\{-\partial_{\mathcal{T}}\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},x)+\mathcal{A}_E\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},x)-\tilde{r}\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},x),\ (x-1)^+-\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},x)\Big\}=0, \tag{3.3.29}$$

on $(0,T_D]\times[0,\infty)$ with initial condition

$$\mathfrak{C}^\star_{\mathbf{A}}(0,x)=(x-1)^+, \qquad x\in[0,\infty). \tag{3.3.30}$$

Proposition 3.2 is due to Pham (cf. [Ph97], [Ph98]), who proved it in greater generality. The proof can be found in his seminal article [Ph98]. Alternatively, we note that Proposition 3.2 could be derived via similar techniques as the ones used in the proof of the upcoming Proposition 3.5 (cf. Appendix C; Section 3.7.3).

[14] Numerous Lévy models considered in the financial literature as well as the model considered in Section 3.5 satisfy these assumptions.

3.3.2.3 PIDE III: Free-Boundary Characterization

Motivated by the theory of early exercise premiums in classical American option settings, we finally aim to derive a free-boundary characterization of the (absolute) tradeability premium $\mathfrak{L}^\star(\cdot)$. We start by collecting in Lemma 3.1 a few useful properties of $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$ that essentially follow from the (strong) Markov property of Lévy processes. A proof of this result is provided in Appendix B (cf. Section 3.7.2).

Lemma 3.1. *The American-type switching option $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$ satisfies the following properties:*

a) For every $\mathcal{T} \in [0, T_D]$, the function $x \mapsto \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x)$ is non-decreasing and convex on $[0, \infty)$.

b) For every $x \in [0, \infty)$, the function $\mathcal{T} \mapsto \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x)$ is non-decreasing on $[0, T_D]$.

c) For every $\mathcal{T} \in [0, T_D]$, we have that

$$|\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, y)| \leq C|x - y|, \qquad \forall x, y \in [0, \infty),$$

with $C = 1$ whenever $\tilde{r} \geq \Phi_Y^{(1)}(1)$.

As in the classical theory of American options, we next decompose the domain $(0, T_D] \times [0, \infty)$ into two regions, the holding region $\mathfrak{D}_h$ and the switching region $\mathfrak{D}_s$. First, combining the results in Lemma 3.1 with Proposition 3.2 ensures that by defining

$$\mathfrak{D}_h := \Big\{(\mathcal{T}, x) \in (0, T_D] \times [0, \infty) :\, \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) > (x-1)^+\Big\}, \tag{3.3.31}$$

$$\mathfrak{D}_s := \Big\{(\mathcal{T}, x) \in (0, T_D] \times [0, \infty) :\, \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) = (x-1)^+\Big\}, \tag{3.3.32}$$

we obtain $\mathfrak{D}_h \dot{\cup} \mathfrak{D}_s = (0, T_D] \times [0, \infty)$. At this point, one should note that nothing has been said about these sets. In fact, while it is easily seen that $\mathfrak{D}_h$ is non-empty, $\mathfrak{D}_s = \emptyset$ could still hold. Looking at Lemma 3.1.*c*) already suggests that this may depend on the sign of $\tilde{r} - \Phi_Y^{(1)}(1)$. Indeed, for $\tilde{r} \leq \Phi_Y^{(1)}(1)$ we obtain that $\mathfrak{D}_s = \emptyset$ and the American-type switching option $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$ reduces to its European counterpart $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$. This follows since, under $\tilde{r} \leq \Phi_Y^{(1)}(1)$, the process $(e^{-\tilde{r}t}E_t)_{t \in [0,T_D]}$ is a ($\mathbb{Q}^{(1)}$-)submartingale.[15] Hence, for $\tilde{r} \leq \Phi_Y^{(1)}(1)$ the tradeability premium is zero and we focus in the following on the case where $\tilde{r} > \Phi_Y^{(1)}(1)$. Here, we show that for any $\mathcal{T} \in (0, T_D]$ there exists a switching boundary $\mathfrak{b}_s(\mathcal{T})$ above which switching to the alternative project is optimal and that it is defined by $\mathfrak{b}_s(\mathcal{T}) := \inf \mathfrak{D}_{s,\mathcal{T}}$, where

$$\mathfrak{D}_{s,\mathcal{T}} := \Big\{x \in [0, \infty) :\, \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) = (x-1)^+\Big\}.$$

To prove the existence of such boundary, we start by proving that, for any $\mathcal{T} \in (0, T_D]$, the set $\mathfrak{D}_{s,\mathcal{T}}$ is non-empty. This is done via similar techniques to the ones used in [FK18] and [DK18]. First, we compute, for $x \in (0, \infty)$ and $f(x) := (x-1)^+$, the instantaneous benefit of waiting to switch, $H(x) := (\mathcal{A}_E f - \tilde{r} f)(x)$, and obtain that

$$H(x) = \Big(\Phi_Y^{(1)}(1)x - \tilde{r}(x-1) - x \int_{\mathbb{R}} (e^y - 1)\Pi_Y(dy)\Big)\mathbb{1}_{\{x \geq 1\}} + \int_{\mathbb{R}} \big(f(xe^y) - f(x)\big)\Pi_Y(dy). \tag{3.3.33}$$

Then, using Peskir's generalized change-of-variable formula (cf. [Pe07]), we obtain that, for any stopping-time $\tau \in \mathfrak{T}_{[0,\mathcal{T}]}$ and $x_0 \in (0, \infty)$,

$$\mathbb{E}^{\mathbb{Q}^{(1)}}_{x_0}\left[e^{-\tilde{r}\tau}\big(E_\tau - 1\big)^+\right] = (x_0 - 1)^+ \;+\; \mathbb{E}^{\mathbb{Q}^{(1)}}_{x_0}\left[\int_0^\tau e^{-\tilde{r}s} H(E_s) ds\right] + \frac{1}{2}\, \mathbb{E}^{\mathbb{Q}^{(1)}}_{x_0}\left[\int_0^\tau e^{-\tilde{r}s} \mathbb{1}_{\{E_{s-}=1,\, E_s=1\}}\, d\ell^1_s(E)\right]. \tag{3.3.34}$$

[15]For $\tilde{r} = \Phi_Y^{(1)}(1)$ it is actually a martingale. However, the submartingale property is for our purpose sufficient (cf. [JYC06]).

Here, $\big(\ell_t^1(E)\big)_{t\in[0,\mathcal{T}]}$ is the local time of $(E_t)_{t\in[0,\mathcal{T}]}$ at the level 1 which is defined, for $t \in [0,\mathcal{T}]$, by means of the equation

$$\big|E_t - 1\big| = \big|E_0 - 1\big| + \int_0^t \operatorname{sgn}(E_{s-} - 1)dE_s + \ell_t^1(E) + \sum_{0<s\le t}\Big(\big|E_s - 1\big| - \big|E_{s-} - 1\big| - \operatorname{sgn}(E_{s-} - 1)\Delta E_s\Big), \quad (3.3.35)$$

where $\operatorname{sgn}(0) := 0$, and $d\ell_s^1(E)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell_s^1(E)$. We claim that Equation (3.3.34) already gives that $\mathfrak{D}_{s,\mathcal{T}} \neq \emptyset$. Indeed, one first obtains that the local time term goes to zero as x_0 becomes large. At the same time, as $x \uparrow \infty$ we have that $H(x) \downarrow -\infty$. This can be seen by combining the condition $\tilde{r} > \Phi_Y^{(1)}(1)$ with the fact that, for any $x > e^1$,

$$\left| \int_{\mathbb{R}} f(xe^y) - f(x) - x(e^y - 1)\Pi_Y(dy) \right| \;\le\; \Pi_Y\big(\{|y| > 1\}\big) < \infty \quad (3.3.36)$$

holds, since for such x we have that

$$\int_{\{|y|\le 1\}} \Big(f(xe^y) - f(x) - x(e^y - 1)\Big)\Pi_Y(dy) \;=\; 0$$

and, for $x \in (0,\infty)$, the function $x \mapsto |f(xe^y) - f(x) - x(e^y - 1)|$ is bounded by 1 (in general, by the strike K), uniformly in y. Due to the lack of time to compensate for the very negative $H(\cdot)$, it is therefore optimal to stop for large x_0 at once. Consequently, $(x_0 - 1)^+ = \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x_0)$ must be true for some $x_0 \in (0,\infty)$. This gives that $\mathfrak{D}_{s,\mathcal{T}} \neq \emptyset$. To see that, for any $\mathcal{T} \in (0, T_D]$, $\mathfrak{b}_s(\mathcal{T}) := \inf \mathfrak{D}_{s,\mathcal{T}}$ gives a boundary with the required properties, we use Lemma 3.1. Indeed, combining Properties *a)* and *c)* of Lemma 3.1 we obtain that whenever $(x-1)^+ = \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x)$ is satisfied for some $x \in [0,\infty)$, we must also have for $y \ge x$ that $(y-1)^+ = \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, y)$. This implies that, for any $\mathcal{T} \in (0, T_D]$, $\mathfrak{D}_{s,\mathcal{T}}$ is an up-connected set and that it can be written as $\mathfrak{D}_{s,\mathcal{T}} = [\inf \mathfrak{D}_{s,\mathcal{T}}, \infty)$, which gives the required properties.

The previous discussion provides an alternative expression for the holding and switching regions, as

$$\mathfrak{D}_h = \big\{(\mathcal{T}, x) \in (0, T_D] \times [0,\infty) : \; x < \mathfrak{b}_s(\mathcal{T})\big\}, \quad (3.3.37)$$

$$\mathfrak{D}_s = \big\{(\mathcal{T}, x) \in (0, T_D] \times [0,\infty) : \; x \ge \mathfrak{b}_s(\mathcal{T})\big\}. \quad (3.3.38)$$

Together with an appropriate smooth-fit property (cf. Appendix B; Section 3.7.2), these results finally lead to the following free-boundary characterization of the (absolute) tradeability premium $\mathfrak{L}^\star(\cdot)$.

Proposition 3.3. *Assume that $\sigma_Y \neq 0$. Then, we have the following properties:*

1. *If $\tilde{r} \le \Phi_Y^{(1)}(1)$, the (absolute) tradeability premium $\mathfrak{L}^\star(\cdot)$ satisfies*

$$\mathfrak{L}^\star(\mathcal{T}, x) = 0, \qquad \forall (\mathcal{T}, x) \in [0, T_D] \times [0,\infty).$$

2. *If $\tilde{r} > \Phi_Y^{(1)}(1)$, the pair $\big(\mathfrak{L}^\star(\cdot), \mathfrak{b}_s(\cdot)\big)$ solves the following free-boundary problem:*

$$-\,\partial_{\mathcal{T}}\mathfrak{L}^\star(\mathcal{T}, x) + \mathcal{A}_E\mathfrak{L}^\star(\mathcal{T}, x) - \tilde{r}\mathfrak{L}^\star(\mathcal{T}, x) = 0, \quad x \in (0, \mathfrak{b}_s(\mathcal{T})),\; \mathcal{T} \in (0, T_D], \quad (3.3.39)$$

subject to the boundary conditions

$$\mathfrak{L}^\star(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) = \mathfrak{b}_s(\mathcal{T}) - 1 - \mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})), \quad \mathcal{T} \in (0, T_D], \quad (3.3.40)$$

$$\partial_x\mathfrak{L}^\star(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) = 1 - \partial_x\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})), \quad \mathcal{T} \in (0, T_D], \quad (3.3.41)$$

$$\mathfrak{L}^\star(\mathcal{T}, 0) = 0, \quad \mathcal{T} \in (0, T_D], \quad (3.3.42)$$

and initial condition

$$\mathfrak{L}^\star(0, x) = 0, \quad x \in (0, \mathfrak{b}_s(\mathcal{T})). \quad (3.3.43)$$

<u>**Remark 3.5.**</u>

i) Proposition 3.3 is of great practical importance. Although we did not obtain an analytical expression for the (absolute) tradeability premium $\mathfrak{L}^\star(\cdot)$, there exist several well-established numerical methods that deal with free-boundary problems in the form of Proposition 3.3. Using such methods, our tradeability valuation problem can be solved for any model that satisfy our (very few) assumptions.

ii) We have just seen that the tradeability premium reduces to zero whenever $\tilde{r} \leq \Phi_Y^{(1)}(1)$, or equivalently whenever $-\big(\Phi_Y(1)-r\big) \leq \Phi_X(1)+\rho\sigma_X\sigma_Y$. From a financial perspective this condition is very intuitive. Indeed, in view of Equation (3.3.15) (and of its derivation), one first obtains that the Laplace exponents $\Phi_X(1)$ and $\Phi_Y(1)$ describe the growth rate of the corresponding processes $(S_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$, i.e. of the initial asset and of the net instantaneous cash-flow generated out of a one-unit investment in the project, respectively. With this understanding, the above condition has the following meaning: It demands that the growth rate of asset $(S_t)_{t\geq 0}$ adjusted for covariance effects across the dynamics of the asset and of the alternative investment exceeds the negative growth, i.e. the loss in terms of the discounted cash-flow level, incurred while waiting to switch to the alternative project.

♦

3.4 Valuing Tradeability: Stochastic Illiquidity Horizon

3.4.1 Generalities

In Section 3.3, we provided a characterization of tradeability premiums when the illiquidity horizon is fully known in advance. Although this characterization already allows for an efficient evaluation of tradeability, starting from a deterministic illiquidity horizon is clearly not a realistic assumption: In practice, agents do not usually know the exact duration of a non-tradeability period and fixing ahead a deterministic illiquidity horizon T_D may seem very simplistic. For this reason, we next extend the previous analysis to the case of a stochastic illiquidity horizon $T_R > 0$. We assume that T_R is exponentially distributed with rate $\vartheta > 0$ and derive tradeability premiums by analyzing randomized versions of the original scenarios:

1. A randomized illiquid scenario, where any attempt to sell the asset $(S_t)_{t\geq 0}$ at time $t \in [0, T_R)$ fails and the investor has to make a new decision at T_R.
2. A randomized liquid scenario, where the tradeability of the investor's asset is guaranteed at any date $t \in [0, T_R]$.

Ideally, we would like to allow for any possible dependency between T_R and the processes $(S_t)_{t\geq 0}$ and $(E_t)_{t\geq 0}$ characterizing the asset replacement. However, dealing with a general stochastic illiquidity horizon can quickly become cumbersome. For this reason, we assume in the sequel that T_R is independent of $(V_t^S)_{t\geq 0}$. Extending our model to allow for a more general dependency structure between T_R and $(V_t^S)_{t\geq 0}$ could be part of future research.

3.4.1.1 Tradeability Premium: Definition

Analyzing both the (randomized) illiquid and liquid scenario can be done via similar arguments to the ones used in their deterministic version and leads to comparable switching options. However, due to the memoryless property of the exponential distribution, the passage of time has no effect on either of the resulting switching options. Consequently, the time-t value of these options is not time-dependent anymore and this leads to the following time-t representations of the investor's trading behavior under stochastic illiquidity horizon:

$$\mathfrak{C}^R_{\mathbf{E}}(S_t, E_t) := \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\big[e^{-rT_R}\left(V^S_{T_R} \vee 0\right)\big] = \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\big[e^{-rT_R} S_{T_R}\left(E_{T_R} - 1\right)^+\big], \tag{3.4.1}$$

$$\mathfrak{C}^R_{\mathbf{A}}(S_t, E_t) := \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\big[e^{-r(T_R\wedge\tau)}\left(V^S_{T_R\wedge\tau} \vee 0\right)\big] = \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\big[e^{-r(T_R\wedge\tau)} S_{T_R\wedge\tau}\left(E_{T_R\wedge\tau} - 1\right)^+\big]. \tag{3.4.2}$$

We therefore identify the (absolute) time-t tradeability premium under stochastic illiquidity horizon $\mathfrak{L}^R(\cdot)$ by means of the relation

$$\mathfrak{L}^R(S_t, E_t) := \mathfrak{C}^R_{\mathbf{A}}(S_t, E_t) - \mathfrak{C}^R_{\mathbf{E}}(S_t, E_t), \tag{3.4.3}$$

and finally note that its relative counterpart is defined accordingly, as

$$\mathfrak{L}^R_{Rel.}(S_t, E_t) := \frac{\mathfrak{L}^R(S_t, E_t)}{\mathfrak{C}^R_{\mathbf{E}}(S_t, E_t)} = \frac{\mathfrak{C}^R_{\mathbf{A}}(S_t, E_t)}{\mathfrak{C}^R_{\mathbf{E}}(S_t, E_t)} - 1.$$

3.4.1.2 Tradeability Premium: Transformation

Following the steps taken in the deterministic version of the problem, we next transform the tradeability valuation equation (3.4.3) into a more tractable expression. First, we introduce, for any $\tau \in \mathfrak{T}_{[0,\infty)} \cup \{\infty\}$, the following notation

$$\mathfrak{C}^R(\tau, S_t, E_t) := \mathbb{E}^{\mathbb{Q}}_{S_t,E_t}\big[e^{-r(T_R\wedge\tau)}\left(V^S_{T_R\wedge\tau} \vee 0\right)\big], \tag{3.4.4}$$

$$\mathfrak{C}^{R,\star}(\tau, E_t) := \mathfrak{C}^R(\tau, 1, E_t), \tag{3.4.5}$$

and note that both $\mathfrak{C}^R_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^R_{\mathbf{A}}(\cdot)$ can be expressed in terms of $\mathfrak{C}^R(\cdot)$ as

$$\mathfrak{C}^R_{\mathbf{E}}(S_t, E_t) = \mathfrak{C}^R(\infty, S_t, E_t) \quad \text{and} \quad \mathfrak{C}^R_{\mathbf{A}}(S_t, E_t) = \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathfrak{C}^R(\tau, S_t, E_t). \tag{3.4.6}$$

Then, conditioning on the random time T_R, allows us to write

$$\mathfrak{C}^R(\tau, S_t, E_t) = S_t \cdot \mathfrak{C}^{R,\star}(\tau, E_t) = S_t \cdot \int_0^\infty \vartheta e^{-\vartheta t_R}\, \mathfrak{C}^\star(t_R \wedge \tau, E_t)\, dt_R, \tag{3.4.7}$$

which implies via Relation (3.4.6) and with $\mathfrak{C}^{R,\star}_{\mathbf{E}}(E_t) := \mathfrak{C}^R_{\mathbf{E}}(1, E_t)$ and $\mathfrak{C}^{R,\star}_{\mathbf{A}}(E_t) := \mathfrak{C}^R_{\mathbf{A}}(1, E_t)$ that

$$\mathfrak{C}^R_{\mathbf{E}}(S_t, E_t) = S_t \cdot \mathfrak{C}^{R,\star}_{\mathbf{E}}(E_t) = S_t \cdot \int_0^\infty \vartheta e^{-\vartheta t_R}\, \mathfrak{C}^\star_{\mathbf{E}}(t_R, E_t)\, dt_R, \tag{3.4.8}$$

$$\mathfrak{C}^R_{\mathbf{A}}(S_t, E_t) = S_t \cdot \mathfrak{C}^{R,\star}_{\mathbf{A}}(E_t) = S_t \cdot \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \int_0^\infty \vartheta e^{-\vartheta t_R}\, \mathfrak{C}^\star(t_R \wedge \tau, E_t)\, dt_R. \tag{3.4.9}$$

Therefore, we focus in the sequel on the valuation of $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^{R,\star}_{\mathbf{A}}(\cdot)$ and solve these valuation problems by relying on results for their deterministic versions $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$.

At this point, we should notice that, in general, the switching options $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$ and $\mathfrak{C}^{R,\star}_{\mathbf{A}}(\cdot)$ may have infinite value for certain parameter choices. To avoid this to happen, we assume in the sequel that the following condition is satisfied

$$\vartheta + \tilde{r} - \Phi_Y^{(1)}(1) > 0. \tag{3.4.10}$$

That this condition indeed rules out infinite values for $\mathfrak{C}_{\mathbf{E}}^{R,\star}(\cdot)$ and $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ can be seen by combining Representation (3.4.9) with Theorem 1 in [Mo02], the submartingale property of the process $(E_t)_{t\geq 0}$ under $\tilde{r} \leq \Phi_Y^{(1)}(1)$, and the well-known representation

$$\mathfrak{C}_{\mathbf{E}}^{\star}(t_R, x) = x e^{-(\tilde{r}-\Phi_Y^{(1)}(1))t_R}\, \tilde{\mathbb{Q}}^{(1)}\big(E_{t_R} \geq 1\big) - e^{-\tilde{r}t_R}\, \mathbb{Q}^{(1)}\big(E_{t_R} \geq 1\big), \tag{3.4.11}$$

where

$$\left.\frac{d\tilde{\mathbb{Q}}^{(1)}}{d\mathbb{Q}^{(1)}}\right|_{\mathcal{F}_t} := e^{Y_t - t\Phi_Y^{(1)}(1)}.$$

3.4.2 OIDE for Tradeability Valuation

We now turn to the derivation of ordinary integro-differential equations (OIDEs) that can be used for tradeability valuation. As earlier, we assume Condition (3.3.14) and focus on the corresponding valuation problems for $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ and $\mathfrak{C}_{\mathbf{E}}^{R,\star}(\cdot)$. We therefore set

$$\mathfrak{L}^{R,\star}(E_t) := \mathfrak{L}^{R}(1, E_t) = \mathfrak{C}_{\mathbf{A}}^{R,\star}(E_t) - \mathfrak{C}_{\mathbf{E}}^{R,\star}(E_t) \tag{3.4.12}$$

and note, as in Remark 3.4., that

$$\mathfrak{L}_{Rel.}^{R}(S_t, E_t) = \mathfrak{L}_{Rel.}^{R}(1, E_t) = \frac{\mathfrak{L}^{R,\star}(E_t)}{\mathfrak{C}_{\mathbf{E}}^{R,\star}(E_t)} = \frac{\mathfrak{C}_{\mathbf{A}}^{R,\star}(E_t)}{\mathfrak{C}_{\mathbf{E}}^{R,\star}(E_t)} - 1.$$

3.4.2.1 OIDE I: Illiquid Scenario

To tackle the illiquid scenario we use Representation (3.4.8) and relevant results for the deterministic valuation problem. Indeed, combining few integrability results with Proposition 3.1 and (strong) Markovian arguments leads to the next proposition. A proof is provided in Appendix C (cf. Section 3.7.3).

Proposition 3.4. *Assume that Conditions (3.4.10) and (3.3.26) hold. Then, the value of the European-type switching option under stochastic illiquidity horizon,* $\mathfrak{C}_{\mathbf{E}}^{R,\star}(\cdot)$*, is continuous on* $[0,\infty)$*,* C^1 *on* $(0,\infty)$ *and solves the ordinary integro-differential equation*

$$\vartheta\left((x-1)^+ - \mathfrak{C}_{\mathbf{E}}^{R,\star}(x)\right) + \mathcal{A}_E \mathfrak{C}_{\mathbf{E}}^{R,\star}(x) - \tilde{r}\mathfrak{C}_{\mathbf{E}}^{R,\star}(x) = 0 \tag{3.4.13}$$

on $(0,\infty)$ *with initial condition*

$$\mathfrak{C}_{\mathbf{E}}^{R,\star}(0) = 0. \tag{3.4.14}$$

3.4.2.2 OIDE II: Liquid Scenario

To deal with the liquid scenario, we start by collecting few properties of the American-type switching option $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ in Lemma 3.2. These results are analogues of the properties presented in Lemma 3.1 and their proof does not substantially differ from the proof provided, for $\mathfrak{C}_{\mathbf{A}}^{\star}(\cdot)$, in Appendix B (cf. Section 3.7.2). Therefore, we only state the results here.

Lemma 3.2. *The following properties hold:*

a) The American-type switching option $x \mapsto \mathfrak{C}_{\mathbf{A}}^{R,\star}(x)$ *is non-decreasing and convex on* $[0,\infty)$.

b) For $\tilde{r} \geq \Phi_Y^{(1)}(1)$ *we have that*

$$|\mathfrak{C}_{\mathbf{A}}^{R,\star}(x) - \mathfrak{C}_{\mathbf{A}}^{R,\star}(y)| \leq |x-y|, \qquad \forall x, y \in [0,\infty).$$

Combining Lemma 3.2 with well-known results for perpetual American options under exponential Lévy models (cf. [Mo02]) allows us to derive the next proposition, which is the analogue of Proposition 3.2 under stochastic illiquidity horizon. This result extends the findings obtained in [Ca98] in the classical Black & Scholes model. A proof is provided in Appendix C (cf. Section 3.7.3).

Proposition 3.5. *Assume that Condition (3.4.10) holds. Then, the value of the American-type switching option under stochastic illiquidity horizon,* $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$, *is continuous on* $[0,\infty)$ *and satisfies the following problem*

$$\vartheta\Big((x-1)^+ - \mathfrak{C}_{\mathbf{A}}^{R,\star}(x)\Big) + \mathcal{A}_E\mathfrak{C}_{\mathbf{A}}^{R,\star}(x) - \tilde{r}\mathfrak{C}_{\mathbf{A}}^{R,\star}(x) = 0, \qquad x \in (0, \mathfrak{b}_s^R), \tag{3.4.15}$$

$$\mathfrak{C}_{\mathbf{A}}^{R,\star}(x) = (x-1)^+, \qquad x \in [\mathfrak{b}_s^R, \infty), \tag{3.4.16}$$

with (unknown) boundary $\mathfrak{b}_s^R > 0$ *and initial condition*

$$\mathfrak{C}_{\mathbf{A}}^{R,\star}(0) = 0. \tag{3.4.17}$$

3.4.2.3 OIDE III: Free-Boundary Characterization

Deriving a free-boundary characterization for the (absolute) tradeability premium under stochastic illiquidity horizon can now be done by relying on the previous results and proofs. First, the proof of Proposition 3.5 reveals that, for $\tilde{r} \leq \Phi_Y^{(1)}(1)$, the American-type switching option $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ reduces to its European counterpart $\mathfrak{C}_{\mathbf{E}}^{R,\star}(\cdot)$. Secondly, combining the latter proof with Lemma 3.2 allows us to derive a representation of the holding and switching regions as

$$\mathfrak{D}_h := \big\{x \in [0,\infty) : \mathfrak{C}_{\mathbf{A}}^{R,\star}(x) > (x-1)^+ \big\} = \big[0, \mathfrak{b}_s^R\big), \tag{3.4.18}$$

$$\mathfrak{D}_s := \big\{x \in [0,\infty) : \mathfrak{C}_{\mathbf{A}}^{R,\star}(x) = (x-1)^+ \big\} = \big[\mathfrak{b}_s^R, \infty\big). \tag{3.4.19}$$

Since the smooth-fit property can be obtained using the same methods as in the deterministic version of the problem (cf. Appendix B; Section 3.7.2), these results finally lead to the following free-boundary characterization of the (absolute) tradeability premium $\mathfrak{L}^{R,\star}(\cdot)$.

Proposition 3.6. *Assume that* $\sigma_Y \neq 0$ *holds. Then, we have the following properties:*

1. *If* $\tilde{r} \leq \Phi_Y^{(1)}(1)$, *the (absolute) tradeability premium* $\mathfrak{L}^{R,\star}(\cdot)$ *satisfies*

$$\mathfrak{L}^{R,\star}(x) = 0, \qquad \forall x \in [0,\infty).$$

2. *If* $\tilde{r} > \Phi_Y^{(1)}(1)$, *the pair* $\big(\mathfrak{L}^{R,\star}(\cdot), \mathfrak{b}_s^R\big)$ *solves the following free-boundary problem:*

$$\mathcal{A}_E\mathfrak{L}^{R,\star}(x) - \big(\tilde{r} + \vartheta\big)\mathfrak{L}^{R,\star}(x) = 0, \quad x \in (0, \mathfrak{b}_s^R), \tag{3.4.20}$$

subject to the boundary conditions

$$\mathfrak{L}^{R,\star}(\mathfrak{b}_s^R) = \mathfrak{b}_s^R - 1 - \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R), \tag{3.4.21}$$

$$\partial_x\mathfrak{L}^{R,\star}(\mathfrak{b}_s^R) = 1 - \partial_x\mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R), \tag{3.4.22}$$

$$\mathfrak{L}^{R,\star}(0) = 0, \tag{3.4.23}$$

<u>**Remark 3.6.**</u>

i) Proposition 3.6 is the analogue of Proposition 3.3 under stochastic illiquidity horizon. As such, it allows for an easy derivation of tradeability values via the application of well-established numerical methods and is therefore of great practical importance.

ii) Recall the financial interpretation of Condition $\tilde{r} \leq \Phi_Y^{(1)}(1)$ from Remark 3.5.ii).

♦

3.5 Numerical Results

To illustrate our approach, we finally derive tradeability premiums by combining the results from Section 3.3 and Section 3.4 with the algorithm developed in [Ma19] and Appendix D (cf. Section 3.7.4).

3.5.1 Model Consideration and Illiquidity Factor

We consider the general asset dynamics defined by (3.2.1)-(3.2.3), i.e. we assume that the initial asset dynamics $(S_t)_{t\geq 0}$ are described (under $\mathbb{Q}$) by

$$dS_t = S_{t-}\bigg(\Phi_X(1)dt + \sigma_X dW_t^X + \int_{\mathbb{R}} (e^y - 1)\tilde{N}_X(dt, dy)\bigg), \tag{3.5.1}$$

and let the cash-flow process $(C_t)_{t\geq 0}$ evolve (under $\mathbb{Q}$) according to (3.2.5) with $(Y_t)_{t\geq 0}$ specified by

$$Y_t := \big(b - \lambda(e^{\varphi} - 1) - \frac{1}{2}\sigma^2\big)t + \sigma W_t^Y + \varphi N_t, \qquad t \geq 0. \tag{3.5.2}$$

As in Section 3.2, $(W_t^X)_{t\geq 0}$ and $(W_t^Y)_{t\geq 0}$ are correlated Brownian motions with correlation $\rho \in [-1, 1]$ and $(N_t)_{t\geq 0}$ denotes a Poisson process with deterministic intensity $\lambda > 0$ and that is independent of the Poisson random measure N_X. We emphasize that (m)any more advanced models could be considered for the dynamics of the cash-flow process $(C_t)_{t\geq 0}$. In particular, the algorithm used in the computation of the liquidity premiums under deterministic illiquidity horizon could be analogously applied under Merton's model as well as under any hyper-exponential jump-diffusion model (cf. [Ma19], [CK11], [CS14]). Neveretheless, we stick for simplicity of the exposition with Dynamics (3.5.2). We will determine (the range of) the relevant parameters in a moment. For now, we just note that $\Phi_Y(1) = b$.

Instead of considering absolute tradeability premiums, we next rely on relative quantities. Additionally, we slightly change our approach: While the relative tradeability premiums $\mathfrak{L}_{Rel.}(\cdot)$ and $\mathfrak{L}_{Rel.}^R(\cdot)$ provide a simple way to evaluate a tradeable asset based on the value of an illiquid equivalent,[16] one is more often interested in the reverse, i.e. in evaluating an illiquid asset given the value of a tradeable equivalent. This motivates the consideration of corresponding time-t illiquidity factors $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}_{Rel.}^R(\cdot)$, defined via

$$\mathfrak{I}_{Rel.}(\mathcal{T}, S_t, E_t) := \mathfrak{I}_{Rel.}(\mathcal{T}, 1, E_t) := \big(1 + \mathfrak{L}_{Rel.}(\mathcal{T}, 1, E_t)\big)^{-1}, \tag{3.5.3}$$

$$\mathfrak{I}_{Rel.}^R(S_t, E_t) := \mathfrak{I}_{Rel.}^R(1, E_t) := \big(1 + \mathfrak{L}_{Rel.}^R(1, E_t)\big)^{-1}. \tag{3.5.4}$$

Our numerical results will focus on these quantities, i.e. we will always express the value of an illiquid asset as percentage of the value of a liquid equivalent. However, as should be clear from (3.5.3) and (3.5.4),

[16] The (time-t) value of a tradeable asset under deterministic and stochastic illiquidity horizon can be readily obtained by multiplying the value of its illiquid equivalent with the factor $\big(1 + \mathfrak{L}_{Rel.}(\mathcal{T}, 1, E_0)\big)$ and $\big(1 + \mathfrak{L}_{Rel.}^R(1, E_0)\big)$, respectively.

relative tradeability premiums and illiquidity factors are dual objects. We will therefore always compute illiquidity factors by means of Relations (3.5.3), (3.5.4) and the tradeability valuation approach discussed in the previous sections.

3.5.2 Parameter Specification

We next specify the parameters in our model: First, we note from the discussion in Section 3.3 and Section 3.4 that dynamics (3.5.1) only influences the relative tradeability premium via the value of its parameters $\Phi_X(1)$, σ_X and ρ. Therefore, (time-t) illiquidity factors can be computed (by means of relative tradeability premiums), once the following parameters are specified: $\mathcal{T}$, ϑ, r, $\Phi_X(1)$, σ_X, ρ, b, σ, φ, λ, C_0.

We determine these parameters by adjusting the parameter choice in [MS86] to current (US-)market data. For instance, all our numerical experiments assume a risk-free rate of 2.25%, which corresponds to a rough average of the US treasury yields with maturity $\mathcal{T} \in \{0.5, 1, 2, 5\}$ as of the end of March 2018.[17] Since our numerical experiments consider the following illiquidity horizons and rates of arrival

$$\mathcal{T} \in \{0.5, 1.5, 2.5, 5\} \qquad \text{and} \qquad \vartheta \in \left\{ \frac{1}{\mathcal{T}} : \ \mathcal{T} \in \{0.5, 1.5, 2.5, 5\} \right\},$$

this risk-free rate seems to be a sensible choice. In analogy to the typical choices made in the option pricing literature, we take the volatility of the initial asset to be either $\sigma_X = 20\%$ or $\sigma_X = 40\%$. Additionally, we set $\Phi_X(1) = 0.005$ and allow this way for a dividend rate of $\delta := r - \Phi_X(1) = 1.75\%$. For the project's cash-flow dynamics, we take three different jump parameters (no jump, negative jump of 15% and negative jump of 30%) and assume that jumps occur on average every 2 years ($\lambda = 0.5$). The volatility of the project is specified by $\sigma = 20\%$. This parameter was already used in [MS86] where it represents the average standard deviation for unlevered equity in the US. The authors obtained it based on the average standard deviation of stocks on the New York Stock Exchange while assuming a debt to value ratio of 1/3 (cf. [MS86]). For the correlation, we take three generic correlation coefficients ($\rho = 0.5$, $\rho = 0$ and $\rho = -0.5$) that were similarly used in [MS86]. Finally, instead of specifying C_0, we express the results in terms of E_0, the expected net present value of the future cash-flow generated out of a one-unit investment in the project, and take $E_0 \in \{0.9, 1.0, 1.1, 1.2\}$. This is only done for the sake of simpler presentation and does not constitute a restriction. Indeed, C_0 can be easily recovered, for each set of parameter, out of E_0 via the relation $E_0 = \frac{C_0}{r-b}$ (cf. Section 3.2 with $\Phi_Y(1) = b$).

3.5.3 Numerical Results: Deterministic Illiquidity Horizon

We first consider the illiquidity factor under deterministic illiquidity horizon, $\mathfrak{I}_{Rel.}(\cdot)$, and derive numerical results by combining Proposition 3.3 with the algorithm developed in [Ma19]. The results are displayed for $b = 0.00$ in Table 3.1 and for $b = -0.04$ in Table 3.2.

As seen from Table 3.1 and Table 3.2, the (relative) tradeability premium substantially depends on the parameter choices and can become very large. Additionally, several properties of the (relative) tradeability premium can be extracted from these tables: As expected, one first sees that the discount for illiquidity, and hence the (relative) tradeability premium, increases with increasing illiquidity horizon $\mathcal{T}$. Moreover, increasing the initial value of the alternative project E_0 (or, equivalently, the initial cash-flow level C_0), increases the discount for illiquidity. Secondly, we notice that diminishing the growth rate of the cash-flow process (i.e. diminishing b) seems to have a positive impact on the value of tradeability. This is intuitively

[17] The following values were extracted from Bloomberg, as of Friday 30 March 2018: 6-month US treasury yield, 1.91%; 1-year US treasury yield, 2.08%; 2-year US treasury yield, 2.27%; 5-year treasury yield, 2.56%.

clear, since reducing the growth rate of the cash-flow process induces a reduction of the project's expected value as time increases. When holding an illiquid asset the investor is forced to keep its position until tradeability (i.e. time T_D) and its final exchange decision will have, in expectation, less value than before.

Table 3.1: Theoretical illiquidity factor, $\mathfrak{I}_{Rel.}(\mathcal{T}, 1, E_0)$, for $b = 0.00$, $\sigma = 0.2$, $\lambda = 0.5$ and $\Phi_X(1) = 0.005$.

		Illiquidity Factor $\mathfrak{I}_{Rel.}(\mathcal{T}, 1, E_0)$								
		No Jump			*Jumps:* $\varphi = \log(0.85)$			*Jumps:* $\varphi = \log(0.7)$		
Parameters		*Correlation* ρ								
	E_0	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$
(1.1.)	0.9	1.000	0.998	0.986	1.000	0.999	0.993	1.000	0.999	0.999
$r = 2.25\%$	1.0	1.000	0.998	0.982	1.000	0.999	0.990	1.000	0.999	0.999
$\sigma_X = 20\%$	1.1	1.000	0.997	0.975	1.000	0.998	0.985	1.000	0.999	0.997
$\mathcal{T} = 0.5$	1.2	1.000	0.996	0.965	1.000	0.998	0.977	1.000	0.999	0.995
(1.2.)	0.9	1.000	0.994	0.966	1.000	0.996	0.975	1.000	0.998	0.991
$r = 2.25\%$	1.0	1.000	0.994	0.958	1.000	0.995	0.968	1.000	0.998	0.987
$\sigma_X = 20\%$	1.1	1.000	0.993	0.947	1.000	0.994	0.960	1.000	0.997	0.983
$\mathcal{T} = 1.5$	1.2	1.000	0.991	0.936	1.000	0.993	0.951	1.000	0.997	0.978
(1.3.)	0.9	1.000	0.990	0.946	1.000	0.992	0.957	1.000	0.995	0.979
$r = 2.25\%$	1.0	1.000	0.989	0.935	1.000	0.991	0.948	1.000	0.994	0.973
$\sigma_X = 20\%$	1.1	1.000	0.987	0.922	1.000	0.989	0.938	1.000	0.994	0.966
$\mathcal{T} = 2.5$	1.2	1.000	0.985	0.909	1.000	0.987	0.927	1.000	0.992	0.959
(1.4.)	0.9	1.000	0.979	0.897	1.000	0.981	0.912	1.000	0.986	0.942
$r = 2.25\%$	1.0	1.000	0.976	0.880	1.000	0.978	0.898	1.000	0.984	0.932
$\sigma_X = 20\%$	1.1	1.000	0.973	0.863	1.000	0.976	0.884	1.000	0.982	0.922
$\mathcal{T} = 5$	1.2	1.000	0.969	0.846	1.000	0.972	0.869	1.000	0.980	0.913
(2.1.)	0.9	1.000	0.998	0.971	1.000	0.999	0.984	1.000	0.999	0.999
$r = 2.25\%$	1.0	1.000	0.998	0.960	1.000	0.999	0.975	1.000	0.999	0.999
$\sigma_X = 40\%$	1.1	1.000	0.997	0.943	1.000	0.998	0.962	1.000	0.999	0.993
$\mathcal{T} = 0.5$	1.2	1.000	0.996	0.921	1.000	0.998	0.945	1.000	0.999	0.985
(2.2.)	0.9	1.000	0.994	0.927	1.000	0.996	0.944	1.000	0.998	0.979
$r = 2.25\%$	1.0	1.000	0.994	0.906	1.000	0.995	0.927	1.000	0.998	0.968
$\sigma_X = 40\%$	1.1	1.000	0.993	0.883	1.000	0.994	0.909	1.000	0.997	0.957
$\mathcal{T} = 1.5$	1.2	1.000	0.991	0.857	1.000	0.993	0.888	1.000	0.997	0.945
(2.3.)	0.9	1.000	0.990	0.883	1.000	0.992	0.906	1.000	0.995	0.951
$r = 2.25\%$	1.0	1.000	0.989	0.857	1.000	0.991	0.884	1.000	0.994	0.937
$\sigma_X = 40\%$	1.1	1.000	0.987	0.830	1.000	0.989	0.862	1.000	0.993	0.922
$\mathcal{T} = 2.5$	1.2	1.000	0.985	0.802	1.000	0.987	0.839	1.000	0.992	0.907
(2.4.)	0.9	1.000	0.979	0.779	1.000	0.981	0.811	1.000	0.986	0.874
$r = 2.25\%$	1.0	1.000	0.976	0.747	1.000	0.978	0.783	1.000	0.984	0.854
$\sigma_X = 40\%$	1.1	1.000	0.973	0.715	1.000	0.976	0.756	1.000	0.982	0.835
$\mathcal{T} = 5$	1.2	1.000	0.969	0.683	1.000	0.972	0.730	1.000	0.980	0.816

Next, looking at the illiquidity factor when varying both the correlation coefficient ρ and the asset's volatility σ_X leads to other interesting properties.[18] First, we note that any increase in correlation leads to a decrease in the discount for illiquidity. However, an increase in the asset's volatility can have various effects on the value of tradeability. Indeed, while an increase in the asset's volatility has no impact on the illiquidity factor, and so on the discount for illiquidity, when the initial asset and the alternative project are uncorrelated ($\rho = 0$),

[18]These properties can be also formally derived by combining the representation of $\mathfrak{L}_{Rel.}(\cdot)$ with Relations (3.3.7) and (3.3.15).

a non-zero correlation can lead to either an increase or a decrease in the discount for illiquidity. In fact, the effect mainly depends on the sign of the correlation coefficient ρ. While an increase in the asset's volatility leads, for $\rho > 0$, to a reduction in the value of tradeability (higher illiquidity factor), the same increase will lead to a higher tradeability premium (lower illiquidity factor), if the correlation coefficient is negative, i.e. if $\rho < 0$.

Table 3.2: Theoretical illiquidity factor, $\mathfrak{I}_{Rel.}(\mathcal{T}, 1, E_0)$, for $b = -0.04$, $\sigma = 0.2$, $\lambda = 0.5$ and $\Phi_X(1) = 0.005$.

		Illiquidity Factor $\mathfrak{I}_{Rel.}(\mathcal{T}, 1, E_0)$								
		No Jump			*Jumps:* $\varphi = \log(0.85)$			*Jumps:* $\varphi = \log(0.7)$		
Parameters		*Correlation* ρ								
	E_0	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$
(1.1.)	0.9	0.986	0.971	0.954	0.993	0.984	0.973	1.000	1.000	1.000
$r = 2.25\%$	1.0	0.982	0.960	0.932	0.990	0.975	0.955	0.999	0.999	0.996
$\sigma_X = 20\%$	1.1	0.975	0.943	0.903	0.985	0.962	0.932	0.997	0.993	0.985
$\mathcal{T} = 0.5$	1.2	0.965	0.921	0.870	0.977	0.945	0.902	0.995	0.985	0.969
(1.2.)	0.9	0.966	0.927	0.878	0.975	0.944	0.905	0.991	0.979	0.961
$r = 2.25\%$	1.0	0.958	0.906	0.843	0.968	0.928	0.876	0.987	0.968	0.941
$\sigma_X = 20\%$	1.1	0.947	0.883	0.805	0.960	0.909	0.845	0.983	0.957	0.921
$\mathcal{T} = 1.5$	1.2	0.936	0.857	0.763	0.951	0.888	0.811	0.978	0.945	0.900
(1.3.)	0.9	0.946	0.883	0.805	0.957	0.906	0.840	0.979	0.951	0.912
$r = 2.25\%$	1.0	0.935	0.857	0.764	0.948	0.884	0.806	0.973	0.937	0.888
$\sigma_X = 20\%$	1.1	0.922	0.830	0.723	0.938	0.862	0.771	0.966	0.922	0.863
$\mathcal{T} = 2.5$	1.2	0.909	0.802	0.680	0.927	0.839	0.735	0.959	0.907	0.839
(1.4.)	0.9	0.897	0.779	0.642	0.912	0.811	0.691	0.942	0.874	0.788
$r = 2.25\%$	1.0	0.880	0.747	0.599	0.898	0.783	0.652	0.932	0.854	0.758
$\sigma_X = 20\%$	1.1	0.863	0.715	0.556	0.884	0.756	0.615	0.922	0.835	0.730
$\mathcal{T} = 5$	1.2	0.846	0.683	0.515	0.869	0.730	0.579	0.913	0.816	0.703
(2.1.)	0.9	0.998	0.971	0.935	0.999	0.984	0.958	1.000	1.000	1.000
$r = 2.25\%$	1.0	0.998	0.960	0.901	0.999	0.975	0.930	0.999	0.999	0.989
$\sigma_X = 40\%$	1.1	0.997	0.943	0.858	0.999	0.962	0.896	0.999	0.993	0.970
$\mathcal{T} = 0.5$	1.2	0.996	0.921	0.820	0.998	0.945	0.854	0.999	0.985	0.944
(2.2.)	0.9	0.994	0.927	0.820	0.996	0.944	0.857	0.998	0.979	0.935
$r = 2.25\%$	1.0	0.994	0.906	0.771	0.995	0.928	0.816	0.998	0.968	0.906
$\sigma_X = 40\%$	1.1	0.993	0.883	0.718	0.994	0.909	0.772	0.997	0.957	0.876
$\mathcal{T} = 1.5$	1.2	0.991	0.857	0.662	0.993	0.888	0.725	0.996	0.945	0.844
(2.3.)	0.9	0.990	0.883	0.715	0.992	0.906	0.763	0.995	0.951	0.862
$r = 2.25\%$	1.0	0.989	0.857	0.663	0.991	0.884	0.717	0.994	0.937	0.828
$\sigma_X = 40\%$	1.1	0.987	0.830	0.609	0.989	0.862	0.671	0.993	0.922	0.793
$\mathcal{T} = 2.5$	1.2	0.985	0.802	0.554	0.987	0.839	0.624	0.992	0.907	0.759
(2.4.)	0.9	0.979	0.779	0.505	0.981	0.811	0.565	0.986	0.874	0.691
$r = 2.25\%$	1.0	0.976	0.747	0.456	0.978	0.783	0.520	0.984	0.854	0.653
$\sigma_X = 40\%$	1.1	0.973	0.715	0.408	0.976	0.756	0.477	0.982	0.835	0.618
$\mathcal{T} = 5$	1.2	0.969	0.683	0.364	0.972	0.730	0.436	0.980	0.816	0.584

Finally, we look at the effect of negative jumps on the size of the tradeability premium. Here, we note that the discount for illiquidity seems to decrease with increasing jump size. Indeed, although negative jumps lead to an abrupt devaluation of the project, they also have a positive effect on the risk-adjusted drift in Dynamics (3.5.2). While these effects neutralize each other in expectation for a fixed time T_D, jumps may substantially affect the value of earlier investments and therefore lead to a decrease in the value of tradeability.

3.5.4 Numerical Results: Stochastic Illiquidity Horizon

We next consider the illiquidity factor under stochastic illiquidity horizon, $\mathfrak{I}^R_{Rel.}(\cdot)$. As shown in Appendix D (cf. Section 3.7.4), the tradeability premium $\mathfrak{L}^{R,\star}(\cdot)$ is now available in semi-closed form. Using these results as well as Relation (3.5.4), we derive corresponding illiquidity factors for $b = 0.00$ and $b = -0.04$. The results are summarized in Table 3.3 and Table 3.4, respectively.

Table 3.3: Theoretical illiquidity factor, $\mathfrak{I}^R_{Rel.}(1, E_0)$, for $b = 0.00$, $\sigma = 0.2$, $\lambda = 0.5$ and $\Phi_X(1) = 0.005$.

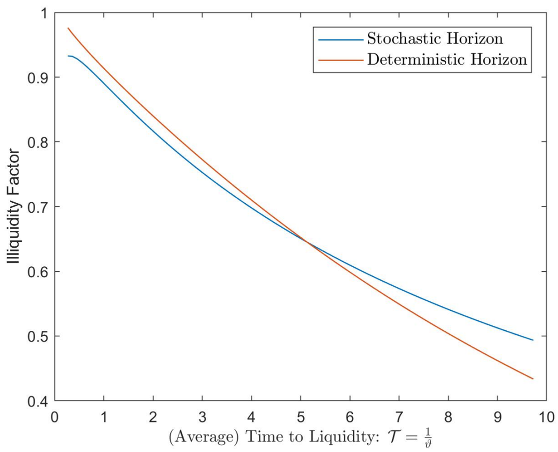
		Illiquidity Factor $\mathfrak{I}^R_{Rel.}(1, E_0)$								
		No Jump			*Jumps:* $\varphi = \log(0.85)$			*Jumps:* $\varphi = \log(0.7)$		
Parameters		*Correlation* ρ								
	E_0	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$
(1.1.)	0.9	1.000	0.999	0.986	1.000	0.977	0.967	1.000	0.937	0.928
$r = 2.25\%$	1.0	1.000	0.999	0.986	1.000	0.977	0.967	1.000	0.937	0.928
$\sigma_X = 20\%$	1.1	1.000	0.999	0.984	1.000	0.993	0.983	1.000	0.978	0.972
$\mathcal{T} = 0.5$	1.2	1.000	0.998	0.977	1.000	0.997	0.984	1.000	0.996	0.991
(1.2.)	0.9	1.000	0.995	0.957	1.000	0.987	0.957	1.000	0.974	0.954
$r = 2.25\%$	1.0	1.000	0.995	0.957	1.000	0.987	0.957	1.000	0.974	0.954
$\sigma_X = 20\%$	1.1	1.000	0.994	0.955	1.000	0.991	0.960	1.000	0.986	0.967
$\mathcal{T} = 1.5$	1.2	1.000	0.993	0.949	1.000	0.993	0.958	1.000	0.993	0.974
(1.3.)	0.9	1.000	0.989	0.928	1.000	0.984	0.934	1.000	0.978	0.944
$r = 2.25\%$	1.0	1.000	0.989	0.928	1.000	0.984	0.934	1.000	0.978	0.944
$\sigma_X = 20\%$	1.1	1.000	0.988	0.926	1.000	0.987	0.935	1.000	0.984	0.951
$\mathcal{T} = 2.5$	1.2	1.000	0.987	0.921	1.000	0.987	0.933	1.000	0.988	0.954
(1.4.)	0.9	1.000	0.970	0.861	1.000	0.968	0.874	1.000	0.968	0.898
$r = 2.25\%$	1.0	1.000	0.970	0.861	1.000	0.968	0.874	1.000	0.968	0.898
$\sigma_X = 20\%$	1.1	1.000	0.969	0.860	1.000	0.969	0.874	1.000	0.970	0.901
$\mathcal{T} = 5$	1.2	1.000	0.969	0.854	1.000	0.969	0.871	1.000	0.972	0.901
(2.1.)	0.9	1.000	0.999	0.963	1.000	0.977	0.949	1.000	0.937	0.915
$r = 2.25\%$	1.0	1.000	0.999	0.963	1.000	0.977	0.949	1.000	0.937	0.915
$\sigma_X = 40\%$	1.1	1.000	0.999	0.957	1.000	0.993	0.964	1.000	0.978	0.962
$\mathcal{T} = 0.5$	1.2	1.000	0.998	0.937	1.000	0.997	0.956	1.000	0.996	0.981
(2.2.)	0.9	1.000	0.995	0.899	1.000	0.987	0.909	1.000	0.974	0.924
$r = 2.25\%$	1.0	1.000	0.995	0.899	1.000	0.987	0.909	1.000	0.974	0.924
$\sigma_X = 40\%$	1.1	1.000	0.994	0.894	1.000	0.991	0.911	1.000	0.986	0.937
$\mathcal{T} = 1.5$	1.2	1.000	0.993	0.877	1.000	0.993	0.902	1.000	0.993	0.942
(2.3.)	0.9	1.000	0.989	0.844	1.000	0.984	0.863	1.000	0.978	0.896
$r = 2.25\%$	1.0	1.000	0.989	0.844	1.000	0.984	0.863	1.000	0.978	0.896
$\sigma_X = 40\%$	1.1	1.000	0.988	0.840	1.000	0.987	0.863	1.000	0.984	0.902
$\mathcal{T} = 2.5$	1.2	1.000	0.987	0.825	1.000	0.987	0.854	1.000	0.988	0.903
(2.4.)	0.9	1.000	0.970	0.732	1.000	0.968	0.761	1.000	0.968	0.814
$r = 2.25\%$	1.0	1.000	0.970	0.732	1.000	0.968	0.761	1.000	0.968	0.814
$\sigma_X = 40\%$	1.1	1.000	0.969	0.729	1.000	0.969	0.760	1.000	0.970	0.816
$\mathcal{T} = 5$	1.2	1.000	0.969	0.717	1.000	0.969	0.752	1.000	0.972	0.814

A brief look at Table 3.3 and Table 3.4 reveals that the (relative) tradeability premium under stochastic illiquidity horizon, $\mathfrak{L}^{R,\star}(\cdot)$, has many similarities to its deterministic equivalent $\mathfrak{L}^{\star}(\cdot)$. Indeed, as its deterministic version, $\mathfrak{L}^{R,\star}(\cdot)$ is an increasing function of the (expected) illiquidity horizon $\mathcal{T} = \frac{1}{\vartheta}$ and a decreasing function in the correlation coefficient ρ. Additionally, reducing the growth rate in the dynamics of the cash-flow process (i.e. reducing b) leads to an increase in the discount for illiquidity (and hence in the value of tradeability). Finally, varying the asset's volatility σ_X may also have various effects on the value of tradeability. Indeed, while an increase in σ_X does not impact the illiquidity factor when $\rho = 0$, the same increase induces, for $\rho > 0$, a reduction and, for $\rho < 0$, an increase in the discount for illiquidity.

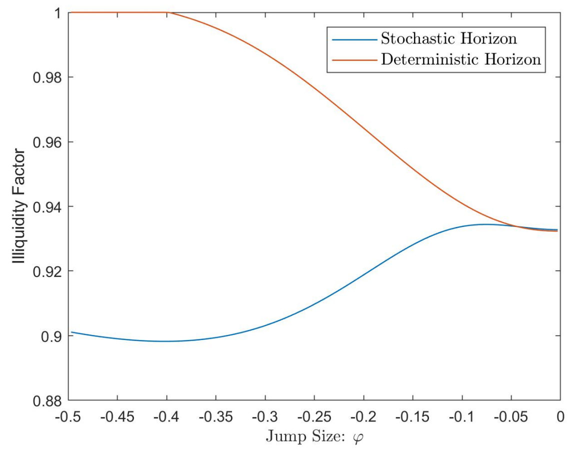
Table 3.4: Theoretical illiquidity factor, $\mathfrak{I}^R_{Rel.}(1, E_0)$, for $b = -0.04$, $\sigma = 0.2$, $\lambda = 0.5$ and $\Phi_X(1) = 0.005$.

Parameters		*Illiquidity Factor $\mathfrak{I}^R_{Rel.}(1, E_0)$*								
		No Jump			*Jumps:* $\varphi = \log(0.85)$			*Jumps:* $\varphi = \log(0.7)$		
		Correlation ρ								
	E_0	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$
(1.1.)	0.9	0.986	0.963	0.933	0.967	0.949	0.926	0.928	0.915	0.899
$r = 2.25\%$	1.0	0.986	0.963	0.933	0.967	0.949	0.926	0.928	0.915	0.899
$\sigma_X = 20\%$	1.1	0.984	0.957	0.922	0.983	0.964	0.938	0.972	0.962	0.948
$\mathcal{T} = 0.5$	1.2	0.977	0.937	0.891	0.984	0.956	0.918	0.991	0.981	0.965
(1.2.)	0.9	0.957	0.899	0.831	0.957	0.909	0.853	0.954	0.924	0.886
$r = 2.25\%$	1.0	0.957	0.899	0.831	0.957	0.909	0.853	0.954	0.924	0.886
$\sigma_X = 20\%$	1.1	0.955	0.894	0.823	0.960	0.911	0.852	0.967	0.937	0.900
$\mathcal{T} = 1.5$	1.2	0.949	0.877	0.793	0.958	0.902	0.834	0.974	0.942	0.901
(1.3.)	0.9	0.928	0.844	0.752	0.934	0.863	0.784	0.944	0.896	0.840
$r = 2.25\%$	1.0	0.928	0.844	0.752	0.934	0.863	0.784	0.944	0.896	0.840
$\sigma_X = 20\%$	1.1	0.926	0.840	0.746	0.935	0.863	0.782	0.951	0.902	0.846
$\mathcal{T} = 2.5$	1.2	0.921	0.825	0.720	0.933	0.854	0.764	0.954	0.903	0.843
(1.4.)	0.9	0.861	0.732	0.611	0.874	0.761	0.651	0.898	0.814	0.728
$r = 2.25\%$	1.0	0.861	0.732	0.611	0.874	0.761	0.651	0.898	0.814	0.728
$\sigma_X = 20\%$	1.1	0.860	0.729	0.606	0.874	0.760	0.648	0.901	0.816	0.730
$\mathcal{T} = 5$	1.2	0.854	0.717	0.586	0.871	0.752	0.634	0.901	0.814	0.724
(2.1.)	0.9	0.999	0.963	0.898	0.977	0.949	0.898	0.937	0.915	0.881
$r = 2.25\%$	1.0	0.999	0.963	0.898	0.977	0.949	0.898	0.937	0.915	0.881
$\sigma_X = 40\%$	1.1	0.999	0.957	0.880	0.993	0.964	0.906	0.978	0.962	0.931
$\mathcal{T} = 0.5$	1.2	0.998	0.937	0.848	0.997	0.956	0.876	0.996	0.981	0.944
(2.2.)	0.9	0.995	0.899	0.761	0.987	0.909	0.792	0.974	0.924	0.844
$r = 2.25\%$	1.0	0.995	0.899	0.761	0.987	0.909	0.792	0.974	0.924	0.844
$\sigma_X = 40\%$	1.1	0.994	0.894	0.748	0.991	0.911	0.788	0.986	0.937	0.857
$\mathcal{T} = 1.5$	1.2	0.994	0.877	0.708	0.993	0.902	0.752	0.993	0.942	0.853
(2.3.)	0.9	0.989	0.844	0.664	0.984	0.863	0.704	0.978	0.896	0.780
$r = 2.25\%$	1.0	0.989	0.844	0.664	0.984	0.863	0.704	0.978	0.896	0.780
$\sigma_X = 40\%$	1.1	0.988	0.840	0.654	0.987	0.863	0.700	0.984	0.902	0.785
$\mathcal{T} = 2.5$	1.2	0.987	0.825	0.617	0.987	0.854	0.674	0.988	0.903	0.778
(2.4.)	0.9	0.970	0.732	0.506	0.968	0.761	0.552	0.968	0.814	0.646
$r = 2.25\%$	1.0	0.970	0.732	0.506	0.968	0.761	0.552	0.968	0.814	0.646
$\sigma_X = 40\%$	1.1	0.970	0.729	0.499	0.969	0.760	0.548	0.970	0.816	0.646
$\mathcal{T} = 5$	1.2	0.969	0.717	0.473	0.969	0.752	0.528	0.972	0.814	0.637

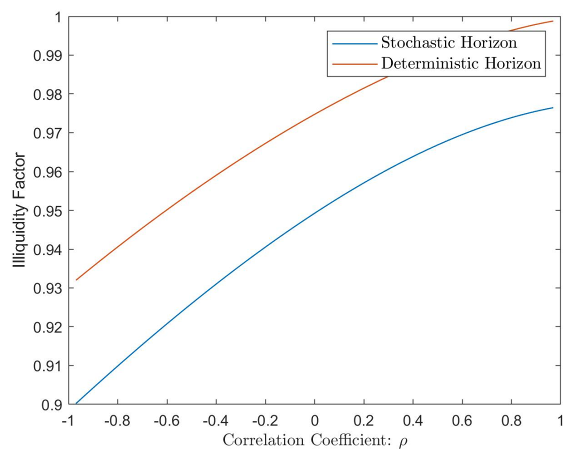
Although $\mathfrak{L}^{R,\star}(\cdot)$ resembles in many ways its deterministic version $\mathfrak{L}^{\star}(\cdot)$, the results in Table 3.3 and Table 3.4 also indicate clear differences between them. Other than for a deterministic illiquidity horizon, the tradeability premium under stochastic illiquidity horizon is no longer monotone in the initial value of the project, E_0. Moreover, the discount for illiquidity is not anymore a monotone function of the jump size φ.



(a) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of $\mathcal{T} = \frac{1}{\vartheta}$.



(b) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of φ.



(c) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of ρ.

Figure 3.1: Illiquidity factor under stochastic illiquidity horizon, $\mathfrak{I}^R_{Rel.}(\cdot)$, and under deterministic illiquidity horizon, $\mathfrak{I}_{Rel.}(\cdot)$, for $\lambda = 0.5$ and as functions of the (expected) illiquidity horizon $\mathcal{T} = \frac{1}{\vartheta}$, the jump size φ, or the correlation coefficient ρ. In Figure 3.1b and Figure 3.1c, we have chosen $\mathcal{T} = \frac{1}{\vartheta} = 0.5$.

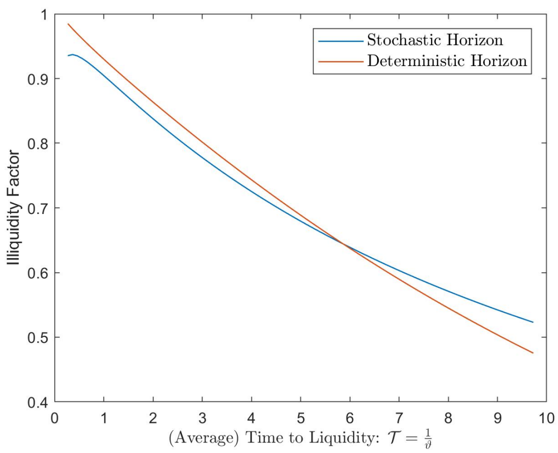
3.5.5 Comparison of the Illiquidity Factors

To finalize the discussion of our numerical results, we provide in Figure 3.1 and Figure 3.2, comparative plots for the illiquidity factor under deterministic and under stochastic illiquidity horizon. Whenever the parameters are not further specified, the following default values are used: $r = 2.25\%$, $\Phi_X(1) = 0.005$, $\sigma_X = 0.2$, $\rho = -0.5$, $b = -0.04$, $\sigma = 0.2$, $\varphi = \log(0.85)$, $E_0 = 1$.

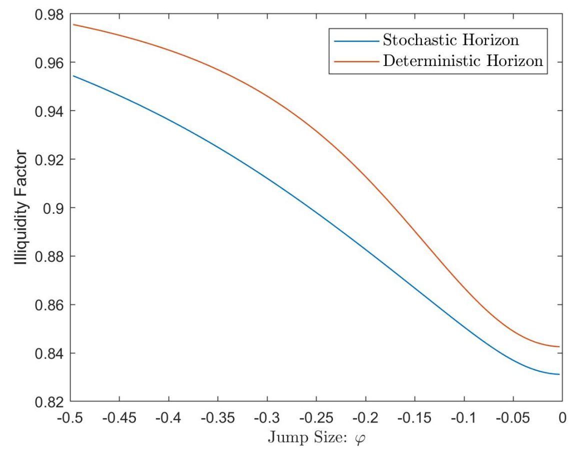
The results in Figure 3.1 and Figure 3.2 confirm several of the properties discussed in Section 3.5.3 and Section 3.5.4. More interestingly, they also show that the tradeability premium under stochastic illiquidity horizon, $\mathfrak{L}^{R,\star}(\cdot)$, may become smaller than its deterministic counterpart. This happens for instance in Figure 3.1a and Figure 3.2a when large (expected) illiquidity horizons $\mathcal{T}$ are considered. In such cases, increasing the uncertainty over the duration of the asset's non-tradeability period raises the asset's value. In particular, this means that "typical market participants" would prefer, under certain parameter specifications, an asset with stochastic illiquidity horizon over an equivalent one with deterministic illiquidity horizon, i.e. the market would exhibit a risk-loving behavior. Although this may be at first surprising, it is a well-documented fact that individuals tend to become risk-loving when confronted with negative events and happen to prefer a gamble over a sure (large) loss. Since illiquidity is, in general, an undesirable feature of an asset, it seems reasonable to observe that individuals may try to avoid large non-tradeability periods by gambling over the illiquidity horizon, i.e. by preferring a stochastic illiquidity horizon over a deterministic illiquidity horizon.

3.6 Conclusion

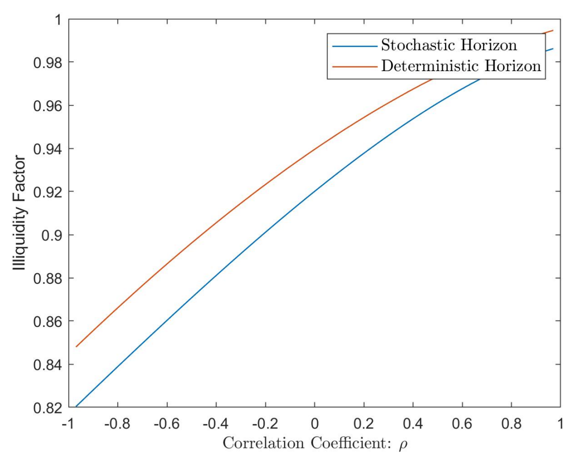
We proposed a new framework to evaluate tradeability and discussed it in the context of exponential Lévy markets. We first introduced our tradeability valuation approach under the simplistic assumption of a deterministic illiquidity horizon and subsequently extended our methods to deal with stochastic illiquidity horizons. Our general framework is linked to the asset replacement problem introduced in [MS86] and allows for a characterization of (individual) tradeability premiums by means of free-boundary problems. The resulting characterizations are of great practical importance, since they allow for a simple computation of tradeability values via the use of well-established numerical schemes. Using such schemes, we illustrated our approach by deriving numerical results and discussing various properties of the tradeability premiums. In particular, we found that, under certain parameter specifications, "typical market participants" may exhibit a risk-loving behavior in the sense that they may prefer an asset with stochastic illiquidity horizon over an equivalent asset with deterministic illiquidity horizon.



(a) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of $\mathcal{T} = \frac{1}{\vartheta}$.



(b) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of φ.



(c) $\mathfrak{I}_{Rel.}(\cdot)$ and $\mathfrak{I}^R_{Rel.}(\cdot)$ as functions of ρ.

Figure 3.2: Illiquidity factor under stochastic illiquidity horizon, $\mathfrak{I}^R_{Rel.}(\cdot)$, and under deterministic illiquidity horizon, $\mathfrak{I}_{Rel.}(\cdot)$, for $\lambda = 1.0$ and as functions of the (expected) illiquidity horizon $\mathcal{T} = \frac{1}{\vartheta}$, the jump size φ, or the correlation coefficient ρ. In Figure 3.2b and Figure 3.2c, we have chosen $\mathcal{T} = \frac{1}{\vartheta} = 1.5$.

3.7 Appendices

3.7.1 Appendix A: Dynamics of $(Y_t)_{t\geq 0}$ under $\mathbb{Q}^{(1)}$

In this appendix, we derive, for any finite time horizon $T > 0$, the dynamics of the Lévy process $(Y_t)_{t\in[0,T]}$ under the particular measure transformation (3.3.6). To this end, we denote by $(X_t^c)_{t\geq 0}$ and $(X_t^d)_{t\geq 0}$ – and $(Y_t^c)_{t\geq 0}$, $(Y_t^d)_{t\geq 0}$ – the continuous and discontinuous parts of $(X_t)_{t\geq 0}$ – and $(Y_t)_{t\geq 0}$ respectively –, i.e. we set

$$X_t^c := b_X t + \sigma_X W_t^X, \qquad X_t^d := X_t - X_t^c, \qquad t \geq 0,$$

– and analogously $Y_t^c := b_Y t + \sigma_Y W_t^Y$, $Y_t^d := Y_t - Y_t^c$, $t \geq 0$. Then, from the independence of the diffusion and jump parts, we note that

$$\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \frac{e^{1\cdot X_t}}{\mathbb{E}^{\mathbb{Q}}\left[e^{1\cdot X_t}\right]} = \frac{e^{1\cdot X_t^c} e^{1\cdot X_t^d}}{\mathbb{E}^{\mathbb{Q}}\left[e^{1\cdot X_t^c}\right]\mathbb{E}^{\mathbb{Q}}\left[e^{1\cdot X_t^d}\right]} =: \frac{d\mathbb{Q}^{(1),c}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} \frac{d\mathbb{Q}^{(1),d}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t}. \tag{A.3.1}$$

Combining this fact with Girsanov's theorem for multidimensional correlated Brownian motion and the properties of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, we obtain, for $m \in \mathbb{N}$, $(\theta_0, \ldots, \theta_{m-1}) \in \mathbb{R}^m$ and $0 \leq t_0 < t_1 < \ldots < t_m \leq T$, that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y_{t_{j+1}} - Y_{t_j})\right\}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_T} \exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y_{t_{j+1}} - Y_{t_j})\right\}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^{(1),c}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_T} \exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y^c_{t_{j+1}} - Y^c_{t_j})\right\}\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y^d_{t_{j+1}} - Y^d_{t_j})\right\}\right] \\
&\quad = \exp\left\{i\sum_{j=0}^{m-1}\theta_j\rho\sigma_X\sigma_Y(t_{j+1}-t_j)\right\}\mathbb{E}^{\mathbb{Q}^{(1),c}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j(\tilde{Y}^c_{t_{j+1}} - \tilde{Y}^c_{t_j})\right\}\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y^d_{t_{j+1}} - Y^d_{t_j})\right\}\right] \\
&\quad = \prod_{j=0}^{m-1}\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{i\theta_j\left[(\tilde{Y}^c_{t_{j+1}} - \tilde{Y}^c_{t_j}) + \rho\sigma_X\sigma_Y(t_{j+1}-t_j)\right]}\right]\prod_{j=0}^{m-1}\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{i\theta_j(Y^d_{t_{j+1}} - Y^d_{t_j})}\right] \\
&\quad = \prod_{j=0}^{m-1}\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{i\theta_j(Y_{t_{j+1}} - Y_{t_j})}\right],
\end{aligned} \tag{A.3.2}$$

where

$$\tilde{Y}_t^c := Y_t^c - \rho\sigma_X\sigma_Y t = b_Y t + \sigma_Y \tilde{W}_t^Y, \qquad \tilde{W}_t^Y := W_t^Y - \rho\sigma_X t,$$

and we have used the fact that $(\tilde{W}_t^Y)_{t\in[0,T]}$ is, under $\mathbb{Q}^{(1),c}$, a Brownian motion – in fact Girsanov's theorem tells us that the processes $(\tilde{W}_t^X)_{t\in[0,T]}$, $\tilde{W}_t^X := W_t^X - \sigma_X t$, and $(\tilde{W}_t^Y)_{t\in[0,T]}$ are correlated Brownian motions under $\mathbb{Q}^{(1)}$. This shows that $(Y_t)_{t\in[0,T]}$ has independent increments under $\mathbb{Q}^{(1)}$.

Showing that $(Y_t)_{t\in[0,T]}$ has stationary increments under $\mathbb{Q}^{(1)}$ is easily done and follows from the identity

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y_{t_{j+1}}-Y_{t_j})\right\}\right]=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\right|_{\mathcal{F}_T}\exp\left\{i\sum_{j=0}^{m-1}\theta_j(Y_{t_{j+1}}-Y_{t_j})\right\}\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\right|_{\mathcal{F}_T}\exp\left\{i\sum_{j=0}^{m-1}\theta_j Y_{(t_{j+1}-t_j)}\right\}\right]=\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp\left\{i\sum_{j=0}^{m-1}\theta_j Y_{(t_{j+1}-t_j)}\right\}\right]. \tag{A.3.3}$$

Finally, it is clear that equivalent measure transformations do not alter both the starting value and the path continuity of processes. Hence, $(Y_t)_{t\in[0,T]}$ is also under $\mathbb{Q}^{(1)}$ càdlàg and satisfies $Y_0=0$. This shows that $(Y_t)_{t\in[0,T]}$ is under $\mathbb{Q}^{(1)}$ again a Lévy process.

Deriving the characteristic exponent of $(Y_t)_{t\in[0,T]}$ under $\mathbb{Q}^{(1)}$ is now easily done using the equation

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{i\theta Y_t}\right]=e^{i\theta\rho\sigma_X\sigma_Y t}\,\mathbb{E}^{\mathbb{Q}^{(1),c}}\left[e^{i\theta\tilde{Y}_t^c}\right]\mathbb{E}^{\mathbb{Q}}\left[e^{i\theta Y_t^d}\right], \tag{A.3.4}$$

which can be derived as in (A.3.2). This gives that the Lévy exponent of $(Y_t)_{t\in[0,T]}$ under $\mathbb{Q}^{(1)}$, $\Psi_Y^{(1)}(\cdot)$, is given by

$$\Psi_Y^{(1)}(\theta)=-i(b_Y+\rho\sigma_X\sigma_Y)\theta+\frac{1}{2}\sigma_Y^2\theta^2+\int_{\mathbb{R}}(1-e^{i\theta y}+i\theta y\mathbb{1}_{\{|y|\leq 1\}})\Pi_Y(dy), \tag{A.3.5}$$

i.e. $(Y_t)_{t\in[0,T]}$ is under $\mathbb{Q}^{(1)}$ an $\mathbf{F}$-Lévy process with triplet $\big(b_Y+\rho\sigma_X\sigma_Y,\sigma_Y^2,\Pi_Y\big)$.

3.7.2 Appendix B: Proofs – Deterministic Illiquidity Horizon

Proof of Proposition 3.1. Due to the discussion preceding Proposition 3.1, we only need to show that $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$ has enough regularity, i.e. in particular that

i) $x\mapsto\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)$ is, for any $\mathcal{T}\in(0,T_D)$, twice continuously differentiable,
ii) $t\mapsto e^{-\tilde{r}t}\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)$ is, for any $x\in(0,\infty)$, continuously differentiable,
iii) $(\mathcal{T},x)\mapsto\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)$ is continuous on $[0,T_D]\times[0,\infty)$.

We start by briefly outlining the proof of i). Since this part does not involve any martingale arguments, we refer the reader for details to [CV05a] and [Vo05]. To see i), one first notices that the European-type option $\mathfrak{C}^\star_{\mathbf{E}}(\cdot)$ can be re-expressed in terms of the function

$$u(\mathcal{T},\xi)=\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\tilde{r}\mathcal{T}}(e^{\xi+Y_{\mathcal{T}}}-1)^+\right],$$

as

$$\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)=\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\tilde{r}\mathcal{T}}(xe^{Y_{\mathcal{T}}}-1)^+\right]=\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\tilde{r}\mathcal{T}}\left(e^{\log(x)+Y_{\mathcal{T}}}-1\right)^+\right]=u\left(\mathcal{T},\log(x)\right). \tag{A.3.6}$$

Therefore, in order to show the smoothness of $x\mapsto\mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T},x)$ it is enough to prove the smoothness of $u(\cdot)$ in the log-moneyness coordinate. To this end, two facts can be combined. First, as noted in [CV05a] and [Vo05], Condition (3.3.26) ensures that Y_t has, for any $t\in[0,T_D]$, a smooth, at least C^2, ($\mathbb{Q}^{(1)}$-)density with derivatives vanishing at infinity. We denote this density in the following by $q_t(\cdot)$. Secondly, setting $\tilde{q}_t(y) := q_t(-y)$, we can rewrite $u(\cdot)$ as a convolution of the form

$$u(\mathcal{T},\xi) = e^{-r\mathcal{T}} \int_{\mathbb{R}} \left(e^{\xi+y}-1\right)^+ q_{\mathcal{T}}(y)\,dy = e^{-r\mathcal{T}} \int_{\mathbb{R}} (e^z-1)^+ \,\tilde{q}_{\mathcal{T}}(\xi - z)\,dz. \tag{A.3.7}$$

Therefore, the decay of $q_{\mathcal{T}}(\cdot)$ and in particular of its derivatives (cf. [CV05a], [Vo05]) allows one to use the dominated convergence theorem to differentiate under the integral sign and to obtain that $x \mapsto \mathfrak{C}^{\star}_{\mathbf{E}}(\mathcal{T},x)$ is twice continuously differentiable.

We now prove ii) using Fourier methods. This approach was similarly used in [CV05a] and relies on a seminal article of Carr and Madan (cf. [CM99]). Recall, for an integrable function $f(\cdot)$, the definition of the Fourier transform, $\mathcal{F}$, and Fourier inverse, $\mathcal{F}^{-1}$,

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} f(y)e^{iy\xi}\,dy, \qquad \mathcal{F}^{-1}f(y) := \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{-i\xi y}\,d\xi,$$

and that both operators can be extended to isometries on the space of square-integrable functions. As noted in i), Condition (3.3.26) ensures that Y_t has, for any $t \in [0,T_D]$, a smooth, C^2, ($\mathbb{Q}^{(1)}$-)density which we will denote again by $q_t(\cdot)$. Therefore, the characteristic function of $Y_{\mathcal{T}}$ at θ, $\chi_{\mathcal{T}}(\theta)$, can be expressed as

$$e^{-\mathcal{T}\Psi_Y^{(1)}(\theta)} = \chi_{\mathcal{T}}(\theta) = \int_{\mathbb{R}} e^{i\theta y} q_{\mathcal{T}}(y)\,dy. \tag{A.3.8}$$

We now consider, for $k \in \mathbb{R}$, the modified call price defined by

$$\mathfrak{c}_{\mathcal{T}}(k) := e^k \int_k^\infty e^{-\tilde{r}\mathcal{T}}(e^y - e^k) q_{\mathcal{T}}(y)\,dy, \tag{A.3.9}$$

and easily see that with $k := \log\left(\frac{K}{x}\right)$, $x \in (0,\infty)$ and $K \in (0,\infty)$, it satisfies that

$$x \cdot \mathfrak{c}_{\mathcal{T}}(k) = x \cdot e^k\, \mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\tilde{r}\mathcal{T}}\left(e^{Y_{\mathcal{T}}} - e^k\right)^+\right] = e^k\, \mathbb{E}^{\mathbb{Q}^{(1)}}_x\left[e^{-\tilde{r}\mathcal{T}}\left(E_{\mathcal{T}} - K\right)^+\right]. \tag{A.3.10}$$

Additionally, we set $\mathfrak{c}^*_{\mathcal{T}}(k) := e^{-\tilde{r}t}\mathfrak{c}_{\mathcal{T}}(k)$. Arguing as in [CM99] one sees that Condition (3.3.14) implies both the integrability and square-integrability of the discounted modified call price $k \mapsto \mathfrak{c}^*_{\mathcal{T}}(k)$. Furthermore one readily derives, using (A.3.9), that

$$\mathcal{F}\mathfrak{c}^*_{\mathcal{T}}(v) = \int_{\mathbb{R}} \mathfrak{c}^*_{\mathcal{T}}(k)e^{ikv}\,dk = \frac{e^{-\tilde{r}T_D}\,\chi_{\mathcal{T}}(v-2i)}{(iv+1)(iv+2)}. \tag{A.3.11}$$

Notice that this expression is clearly differentiable with respect to t and that one obtains

$$\partial_t \mathcal{F}\mathfrak{c}^*_{\mathcal{T}}(v) = \frac{e^{-\tilde{r}T_D}\,\chi_{\mathcal{T}}(v-2i)\,\Psi_Y^{(1)}(v-2i)}{(iv+1)(iv+2)}. \tag{A.3.12}$$

From the Lévy-Khintchine formula/representation, one additionally sees that $\Psi_Y^{(1)}(v-2i) = \mathcal{O}(|v|^2)$ (as $|v| \to \infty$) – hence at ∞ the denominator compensates $\Psi_Y^{(1)}(v-2i)$. Combining these arguments with the fact that, under (3.3.26),

$$|\chi_{\mathcal{T}}(z)| \le C(\mathcal{T})\exp(-c(\mathcal{T})|z|^\gamma) \quad \text{for some} \quad \gamma > 0^{19} \quad \text{and “constants”} \quad C(\mathcal{T}), c(\mathcal{T}) > 0 \tag{A.3.13}$$

[19] $\gamma = 2$ if $\sigma \neq 0$ and $\gamma = \alpha$ if $\sigma = 0$ and the second condition is satisfied. This was already noted in [Vo05] (cf. [Sa99]).

and, in particular, that $\mathcal{T} \mapsto C(\mathcal{T})$, $\mathcal{T} \mapsto c(\mathcal{T})$ can be chosen to be continuous (by the continuity of $\mathcal{T} \mapsto \chi_{\mathcal{T}}(z)$), tells us that (A.3.12) is in any case dominated (locally in $\mathcal{T}$) by an integrable function that does not have any $\mathcal{T}$-dependency.[20] Finally, this allows us to use the dominated convergence theorem in order to conclude that

$$\partial_t \mathfrak{c}^*_{\mathcal{T}}(k) = \partial_t \mathcal{F}^{-1}\mathcal{F}\mathfrak{c}^*_{\mathcal{T}}(k) = \mathcal{F}^{-1}\partial_t \mathcal{F}\mathfrak{c}^*_{\mathcal{T}}(k), \tag{A.3.14}$$

which shows, in particular by means of Relation (A.3.10) with $K = 1$, that $t \mapsto e^{-\tilde{r}t}\mathfrak{C}^{\star}_{\mathbf{E}}(\mathcal{T}, x)$ is for any $x \in (0, \infty)$ differentiable. The continuity of the derivative is easily seen from (A.3.14) and (A.3.12) and the dominated convergence theorem, by noting that $t \mapsto \chi_{\mathcal{T}}(v - 2i)$ is continuous (recall that $\mathcal{T} = T_D - t$).

Finally, iii) is a direct consequence of Relation (A.3.10) and the continuity of $(\mathcal{T}, k) \mapsto \mathfrak{c}_{\mathcal{T}}(k)$, which follows again from (A.3.11) by means of Fourier inversion and the dominated convergence theorem. This finalizes the proof. □

Proof of Lemma 3.1. The first part of *a*), i.e. the non-decreasing property follows directly from the path properties of exponential Lévy models. As this is easily proved, we focus on showing the convexity of the American-type option. To start, let $\mathcal{T} \in [0, T_D]$ be arbitrary but fixed. We define, for any initial value $x \in [0, \infty)$ and any stopping time $\tau \in \mathfrak{T}_{[0,\mathcal{T}]}$, the two value functions $V(\cdot)$ and $V^*(\cdot)$ by

$$V(\tau, x) := \mathbb{E}^{\mathbb{Q}^{(1)}}_x \left[e^{-\tilde{r}\tau} \big(E_\tau - 1\big)^+ \right] \tag{A.3.15}$$

and

$$V^*(x) := \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} V(\tau, x), \tag{A.3.16}$$

and note that $V^*(x) = \mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, x)$. Given two initial values x_1 and x_2 and an arbitrary $\lambda \in [0, 1]$, we set $\tilde{x} := \lambda x_1 + (1 - \lambda)x_2$ and fix some $\epsilon > 0$. By definition of $V^*(\cdot)$, we can find a stopping time τ_ϵ satisfying $V^*(\tilde{x}) \le V(\tau_\epsilon, \tilde{x}) + \epsilon$. Furthermore, from the (strong) Markov property of $(E_t)_{t\in[0,T_D]}$ and the properties of the pay-off function, we have that

$$V(\tau_\epsilon, \tilde{x}) \le \lambda V(\tau_\epsilon, x_1) + (1 - \lambda)V(\tau_\epsilon, x_2), \tag{A.3.17}$$

which implies that

$$V^*(\tilde{x}) \le V(\tau_\epsilon, \tilde{x}) + \epsilon \le \lambda V(\tau_\epsilon, x_1) + (1 - \lambda)V(\tau_\epsilon, x_2) + \epsilon \le \lambda V^*(x_1) + (1 - \lambda)V^*(x_2) + \epsilon. \tag{A.3.18}$$

Since ϵ was arbitrary, this gives the convexity of the American-type option.

Property *b*) follows directly by noting that, for $0 \le \mathcal{T}_1 \le \mathcal{T}_2 \le T_D$, any stopping time $\tau \in \mathfrak{T}_{[0,\mathcal{T}_1]}$ also satisfies $\tau \in \mathfrak{T}_{[0,\mathcal{T}_2]}$. Therefore, we are left with Part *c*). To prove this last part, we use the (strong) Markov property of $(E_t)_{t\in[0,T_D]}$ as well as the property that, for $x, y \in [0, \infty)$, $\big|(x-1)^+ - (y-1)^+\big| \le |x - y|$ holds. We then obtain, for a fixed $\mathcal{T} \in [0, T_D]$, that

$$\begin{aligned}
&\Big| \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}^{(1)}}_x \left[e^{-\tilde{r}\tau} \big(E_\tau - 1\big)^+ \right] - \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}^{(1)}}_y \left[e^{-\tilde{r}\tau} \big(E_\tau - 1\big)^+ \right] \Big| \\
&\qquad \le \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \Big| \mathbb{E}^{\mathbb{Q}^{(1)}}_x \left[e^{-\tilde{r}\tau} \big(E_\tau - 1\big)^+ \right] - \mathbb{E}^{\mathbb{Q}^{(1)}}_y \left[e^{-\tilde{r}\tau} \big(E_\tau - 1\big)^+ \right] \Big| \\
&\qquad \le \big|x - y\big| \cdot \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}^{(1)}} \left[e^{-(\tilde{r} - \Phi^{(1)}_Y(1))\tau} e^{Y_\tau - \tau\Phi^{(1)}_Y(1)} \right].
\end{aligned} \tag{A.3.19}$$

[20] It suffices to take, for a given (compact) $\mathcal{T}$-neighborhood U, $C^\star := \max_{t\in U} C(t)$ and $c^\star := \min_{t\in U} c(t)$ in (A.3.13).

Since the process $\left(e^{Y_t - t\Phi_Y^{(1)}(1)}\right)_{t\in[0,T_D]}$ is known to be a ($\mathbb{Q}^{(1)}$-)martingale, we can take

$$C := \begin{cases} 1, & \text{if } \tilde{r} \geq \Phi_Y^{(1)}(1), \\ e^{-(\tilde{r}-\Phi_Y^{(1)}(1))\mathcal{T}}, & \text{otherwise,} \end{cases}$$

and obtain from (A.3.19) that

$$|\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},x) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T},y)| \leq C|x-y|.$$

□

Proof of the smooth-fit property in Proposition 3.3. This part provides a proof of Equation (3.3.41), i.e. we show that, for all $\mathcal{T} \in (0, T_D]$, we have

$$\partial_x \mathfrak{L}^\star(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) = 1 - \partial_x \mathfrak{C}^\star_{\mathbf{E}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})). \tag{A.3.20}$$

For this equation to hold, it is sufficient to have that, for any $\mathcal{T} \in (0, T_D]$, the function $x \mapsto \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x)$ is in $\mathfrak{b}_s(\mathcal{T})$ differentiable with $\partial_x \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) = 1$. We show that this is true.

First, we recall that for a Lévy process $(Z_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed level $z \in \mathbb{R}$ is said to be regular for (z, ∞), if we have that

$$\mathbb{P}_z(\tau_z^+ = 0) = 1,$$

where τ_z^+ is given by

$$\tau_z^+ := \inf\{t \geq 0 : \ Z_t \in (z, \infty)\},$$

and we set as usual $\inf \emptyset = \infty$. As noted for instance in [Ky06], Theorem 6.5, any Lévy process of infinite variation has the particularity that the point 0 is regular for the interval $(0, \infty)$. Since we have assumed that $\sigma_Y \neq 0$, the ($\mathbb{Q}^{(1)}$-)Lévy process $(Y_t)_{t\geq 0}$ has clearly infinite variation (c.f. [Sa99], [Ap09]). Therefore, it suffices to show that the regularity of 0 for $(0, \infty)$ and $(Y_t)_{t\geq 0}$ implies the smooth-fit property of $\mathfrak{C}^\star_{\mathbf{A}}(\cdot)$. We show it by adapting the proof of Theorem 4.1. in [LM11]:
Let us fix $\mathcal{T} \in (0, T_D]$. We start by noting that

$$\lim_{h\downarrow 0} \frac{\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} = 1. \tag{A.3.21}$$

This directly follows since any $x \geq \mathfrak{b}_s(\mathcal{T})$ satisfies that $\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) = x - 1$. Therefore, we only need to show that

$$\lim_{h\uparrow 0} \frac{\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} = 1. \tag{A.3.22}$$

First, we obtain from $\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) = (\mathfrak{b}_s(\mathcal{T}) - 1)^+$ and $\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, x) \geq (x-1)^+$ that, for any $h < 0$,

$$\frac{\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} \geq \frac{(\mathfrak{b}_s(\mathcal{T}) + h - 1)^+ - (\mathfrak{b}_s(\mathcal{T}) - 1)^+}{h}.$$

This gives that

$$\liminf_{h\uparrow 0} \frac{\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} \geq 1. \tag{A.3.23}$$

To show that

$$\limsup_{h\uparrow 0} \frac{\mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^\star_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} \leq 1, \tag{A.3.24}$$

we consider, for $h < 0$, the optimal stopping problem related to $\mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h)$: First, we define the stopping time

$$\begin{aligned}\tau_h &:= \inf\{t \in [0, \mathcal{T}) : \; (\mathfrak{b}_s(\mathcal{T}) + h)\, e^{Y_t} \geq \mathfrak{b}_s(\mathcal{T})\} \\ &= \inf\left\{t \in [0, \mathcal{T}) : \; Y_t \geq \log\left(\frac{\mathfrak{b}_s(\mathcal{T})}{\mathfrak{b}_s(\mathcal{T}) + h}\right)\right\}\end{aligned} \tag{A.3.25}$$

and note from the regularity of 0 for the set $(0, \infty)$ that $\tau_h \to 0$ a.s. when $h \uparrow 0$. This can be seen by the following argument: On the almost sure set $\{\tau_0^+ = 0\}$, we can find for any $t_0 \in (0, \mathcal{T})$ a point $u \in [0, t_0]$ such that $Y_u > 0$. Then, taking $h < 0$ small enough (i.e. near enough to zero) gives that $Y_u > \log\left(\frac{\mathfrak{b}_s(\mathcal{T})}{\mathfrak{b}_s(\mathcal{T})+h}\right)$. Consequently, $\lim_{h\uparrow 0} \tau_h \leq t_0$ a.s. and from the arbitrariness of $t_0 \in (0, \mathcal{T})$ this already gives that $\lim_{h\uparrow 0} \tau_h = 0$. Next, noting that

$$\mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T})) \geq \mathbb{E}^{\mathbb{Q}^{(1)}}_{\mathfrak{b}_s(\mathcal{T})}\left[e^{-\tilde{r}\tau_h}\,(E_{\tau_h} - 1)^+\right]$$

and combining this inequality with the optimality of the stopping time τ_h for the starting value $\mathfrak{b}_s(\mathcal{T}) + h$ gives, for $h < 0$, that

$$\begin{aligned}\frac{\mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} &= \frac{\mathbb{E}^{\mathbb{Q}^{(1)}}_{\mathfrak{b}_s(\mathcal{T})+h}\left[e^{-\tilde{r}\tau_h}\,(E_{\tau_h} - 1)^+\right] - \mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} \\ &\leq \mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{-\tilde{r}\tau_h}\,\frac{\left((\mathfrak{b}_s(\mathcal{T}) + h)e^{Y_{\tau_h}} - 1\right)^+ - \left(\mathfrak{b}_s(\mathcal{T})e^{Y_{\tau_h}} - 1\right)^+}{h}\right].\end{aligned} \tag{A.3.26}$$

Since $x \mapsto (x - 1)^+$ is continuously differentiable in a neighbourhood of $\mathfrak{b}_s(\mathcal{T})$, we have that

$$\lim_{h\uparrow 0} \frac{\left((\mathfrak{b}_s(\mathcal{T}) + h)e^{Y_{\tau_h}} - 1\right)^+ - \left(\mathfrak{b}_s(\mathcal{T})e^{Y_{\tau_h}} - 1\right)^+}{h} = 1. \tag{A.3.27}$$

Finally, using Lemma 3.1.c) allows us to apply the dominated convergence theorem, to obtain that

$$\limsup_{h\uparrow 0} \frac{\mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}) + h) - \mathfrak{C}^{\star}_{\mathbf{A}}(\mathcal{T}, \mathfrak{b}_s(\mathcal{T}))}{h} \leq 1,$$

which gives the result. $\square$

3.7.3 Appendix C: Proofs – Stochastic Illiquidity Horizon

Proof of Proposition 3.4. First, we note that the continuity of $x \mapsto \mathfrak{C}^{R,\star}_{\mathbf{E}}(x)$ on $[0, \infty)$ follows from the dominated convergence theorem, by combining Condition (3.4.10) with Representations (3.4.8) and (3.4.11). Additionally, the continuity of $x \mapsto \partial_x \mathfrak{C}^{R,\star}_{\mathbf{E}}(x)$ on $(0, \infty)$ follows analogously using (3.4.8), the continuity of $x \mapsto \mathfrak{C}^{\star}_{\mathbf{E}}(t_R, x)$ for all $t_R > 0$, and the inequality

$$\left|\mathfrak{C}^{\star}_{\mathbf{E}}(t_R, x) - \mathfrak{C}^{\star}_{\mathbf{E}}(t_R, y)\right| \leq e^{-(\tilde{r} - \Phi^{(1)}_Y(1))t_R}|x - y|, \qquad \forall x, y \in (0, \infty).$$

Therefore, we are left with the proof of Equations (3.4.13), (3.4.14). Here, we start by re-considering the $\tilde{r}$-killed version of $(E_t)_{t\geq 0}$, $(\bar{E}_t)_{t\geq 0}$, i.e. the process whose transition probabilities are given by

$$\mathbb{Q}^{(1)}_x\left(\bar{E}_t \in A\right) = \mathbb{E}^{\mathbb{Q}^{(1)}}_x\left[e^{-\tilde{r}t}\,\mathbb{1}_A(E_t)\right], \tag{A.3.28}$$

and identify, without loss of generality, its cemetery state with $\partial \equiv 0$. We then re-express $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$ as solution to an optimal stopping problem: We view the stochastic illiquidity horizon T_R as jump time of a corresponding Poisson process[21] $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$ and consider, for any $z = (n, x) \in \mathbb{N}_0 \times [0, \infty)$, the (strong) Markov process $(Z_t)_{t\geq 0}$ defined by means of $Z_t := (n + N_t, \bar{E}_t)$, $\bar{E}_0 = x$, on the state domain $\mathcal{D} := \mathbb{N}_0 \times [0, \infty)$. Then, $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$ can be equivalently written as

$$\mathfrak{C}^{R,\star}_{\mathbf{E}}(x) = \widetilde{V}_E\big((0, x)\big), \tag{A.3.29}$$

where, for $z = (n, x) \in \mathcal{D}$, the value function $\widetilde{V}_E(\cdot)$ is defined, under the measure $\mathbb{Q}^{(1),Z}_z$ having initial distribution $Z_0 = z$, by

$$\widetilde{V}_E(z) := \mathbb{E}^{\mathbb{Q}^{(1),Z}}_z \left[G\big(Z_{\tau_{\mathcal{S}}}\big) \right], \qquad G(z) := (x - 1)^+, \tag{A.3.30}$$

and $\tau_{\mathcal{S}} := \inf\{t \geq 0 : Z_t \in \mathcal{S}\}$, $\mathcal{S} := \big(\mathbb{N} \times (0, \infty)\big) \cup \big(\mathbb{N}_0 \times \{0\}\big)$, is a stopping time that is $\mathbb{Q}^{(1),Z}_z$-almost surely finite for any $z = (n, x)$.[22] Furthermore, the stopping domain $\mathcal{S}$ forms (under an appropriate product-metric) a closed set in $\mathcal{D}$.[23] Therefore, standard arguments based on the strong Markov property of $(Z_t)_{t\geq 0}$ (cf. [PS06]) imply that $\widetilde{V}_E(\cdot)$ solves the following problem

$$\mathcal{A}_Z \widetilde{V}_E(z) = 0, \qquad \text{on } \mathcal{D} \setminus \mathcal{S}, \tag{A.3.31}$$

$$\widetilde{V}_E(z) = G(z), \qquad \text{on } \mathcal{S}, \tag{A.3.32}$$

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t\geq 0}$. To complete the proof, it therefore suffices to note that (for any suitable function $V : \mathcal{D} \to \mathbb{R}$) the infinitesimal generator $\mathcal{A}_Z$ can be re-expressed as

$$\begin{aligned} \mathcal{A}_Z V\big((n, x)\big) &= \mathcal{A}^n_N V\big((n, x)\big) + \mathcal{A}^x_{\bar{E}} V\big((n, x)\big) \\ &= \vartheta \left(V\big((n + 1, x)\big) - V\big((n, x)\big) \right) + \mathcal{A}^x_E V\big((n, x)\big) - \tilde{r} V\big((n, x)\big), \end{aligned} \tag{A.3.33}$$

where $\mathcal{A}_N$ denotes the infinitesimal generator of the Poisson process $(N_t)_{t\geq 0}$ and the notation $\mathcal{A}^n_N$, $\mathcal{A}^x_{\bar{E}}$, and $\mathcal{A}^x_E$ is used to indicate that the generators are applied to n and x respectively. Indeed, recovering $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$ via (A.3.29) while noting Relation (A.3.33) and the fact that for any $x \in [0, \infty)$ we have

$$\widetilde{V}_E\big((1, x)\big) = G\big((1, x)\big) = (x - 1)^+ \tag{A.3.34}$$

finally gives the claim. □

Proof of Proposition 3.5. To start, we note that, under $\tilde{r} \leq \Phi^{(1)}_Y(1)$, the American-type switching option $\mathfrak{C}^{R,\star}_{\mathbf{A}}(\cdot)$ reduces to its European counterpart $\mathfrak{C}^{R,\star}_{\mathbf{E}}(\cdot)$. As earlier, this is a direct consequence of the fact that the process $\big(e^{-\tilde{r}t} E_t\big)_{t\geq 0}$ then becomes a ($\mathbb{Q}^{(1)}$-)submartingale. In this case, the result directly follows via Proposition 3.4, with $\mathfrak{b}^R_s = \infty$.

[21] Our assumptions on T_R clearly imply that the Poisson process is independent of $(\bar{E}_t)_{t\geq 0}$.

[22] The finiteness of this stopping time directly follows from the properties (e.g. finiteness of the first moment) of the exponential distribution of any intensity $\vartheta > 0$.

[23] We note that several choices of a product-metric on $\mathcal{D}$ give the closedness of the set $\mathcal{S}$. In particular, one may choose on $\mathbb{N}_0$ the following metric

$$d_{\mathbb{N}_0}(m, n) := \begin{cases} 1 + |2^{-m} - 2^{-n}|, & m \neq n, \\ 0, & m = n, \end{cases}$$

and consider the product-metric on $\mathcal{D}$ obtained by combining $d_{\mathbb{N}_0}(\cdot, \cdot)$ on $\mathbb{N}_0$ with the Euclidean metric on $[0, \infty)$.

For $\tilde{r} > \Phi_Y^{(1)}(1)$, we first note that Theorem 1 in [Mo02] implies the existence of a finite optimal stopping boundary $\mathfrak{b}_s^R > 0$. Indeed, this follows by combining Lemma 3.2 with the fact that, with

$$\mathfrak{C}_{\mathbf{A}}^{\infty,\star}(x) := \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathfrak{C}^{\star}(\tau, x),$$

$$\big\{x \in [0,\infty) : \mathfrak{C}_{\mathbf{A}}^{\infty,\star}(x) = (x-1)^+\big\} \subseteq \big\{x \in [0,\infty) : \mathfrak{C}_{\mathbf{A}}^{R,\star}(x) = (x-1)^+\big\}$$

and by arguing as in Section 3.3.2.3. Therefore, by viewing the stochastic illiquidity horizon T_R as jump time of a corresponding Poisson process $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$, we can re-express our optimal stopping problem in the following form: We consider, for any $z = (n,x) \in \mathbb{N}_0 \times [0,\infty)$, the (strong) Markov process $(Z_t)_{t\geq 0}$ defined by means of $Z_t := (n + N_t, \bar{E}_t)$, $\bar{E}_0 = x$, on the state domain $\mathcal{D} := \mathbb{N}_0 \times [0,\infty)$ and identify again its cemetery state with $\partial \equiv 0$. Then, we note that

$$\mathfrak{C}_{\mathbf{A}}^{R,\star}(x) = \widetilde{V}_A\big((0,x)\big), \tag{A.3.35}$$

where, for $z = (x,n) \in \mathcal{D}$, the value function $\widetilde{V}_A(\cdot)$ is defined, under the measure $\mathbb{Q}_z^{(1),Z}$ having initial distribution $Z_0 = z$, by

$$\widetilde{V}_A(z) := \mathbb{E}_z^{\mathbb{Q}^{(1),Z}}\left[G\big(Z_{\tau_{\mathcal{S}}}\big)\right], \qquad G(z) := (x-1)^+, \tag{A.3.36}$$

and $\tau_{\mathcal{S}} := \inf\{t \geq 0 : Z_t \in \mathcal{S}\}$, $\mathcal{S} := \big(\mathbb{N} \times (0,\infty)\big) \cup \big(\mathbb{N}_0 \times \{0\}\big) \cup \big(\{0\} \times [\mathfrak{b}_s^R, \infty)\big)$ is a stopping time that is $\mathbb{Q}_z^{(1),Z}$-almost surely finite for any $z = (n,x)$. Furthermore, the stopping domain $\mathcal{S}$ forms (under an appropriate product-metric) a closed set in $\mathcal{D}$.[24] Therefore, standard arguments based on the strong Markov property of $(Z_t)_{t\geq 0}$ (cf. [PS06]) imply that $\widetilde{V}_A(\cdot)$ solves the following problem

$$\mathcal{A}_Z \widetilde{V}_A(z) = 0, \qquad \text{on } \mathcal{D} \setminus \mathcal{S}, \tag{A.3.37}$$

$$\widetilde{V}_A(z) = G(z), \qquad \text{on } \mathcal{S}, \tag{A.3.38}$$

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t\geq 0}$. To complete the proof, we therefore argue as in the proof of Proposition 3.4, i.e. we recover $\mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ via (A.3.35) and combine Relation (A.3.33) with the fact that for any $x \in [0,\infty)$ we have

$$\widetilde{V}_A\big((1,x)\big) = G\big((1,x)\big) = (x-1)^+. \tag{A.3.39}$$

Since Equation (3.4.17) is naturally satisfied, this leads to the required problem. The continuity of the function $x \mapsto \mathfrak{C}_{\mathbf{A}}^{R,\star}(\cdot)$ directly follows from its convexity (cf. Lemma 3.2). Therefore, the proof is complete. □

3.7.4 Appendix D: Derivation of $\mathfrak{L}^{R,\star}(\cdot)$

In this appendix, we briefly derive a semi-analytical solution to the free-boundary problem of Proposition 3.6, when the dynamics of $(S_t)_{t\geq 0}$ and $(E_t)_{t\geq 0}$ are given by (3.5.1) and (3.2.5), (3.5.2) and assuming non-positive jumps, i.e. $\varphi \leq 0$. This is used to obtain numerical results in Section 3.5.4.

To start, we first note that, under the given dynamics and with $\tilde{b} := b + \rho\sigma_X\sigma$, the free-boundary problem reads:

1. If $\tilde{r} \leq \tilde{b}$, the (absolute) tradeability premium $\mathfrak{L}^{R,\star}(\cdot)$ satisfies

$$\mathfrak{L}^{R,\star}(x) = 0, \qquad \forall x \in [0,\infty).$$

[24] As earlier, this property can be obtained under the product-metric considered in Footnote 23.

2. If $\tilde{r} > \tilde{b}$, the pair $\left(\mathfrak{L}^{R,\star}(\cdot), \mathfrak{b}_s^R\right)$ solves the following free-boundary problem:

$$\frac{1}{2}\sigma^2 x^2 \partial_x^2 \mathfrak{L}^{R,\star}(x) + \big(\tilde{b} - \lambda(e^{\varphi} - 1)\big) x \partial_x \mathfrak{L}^{R,\star}(x) + \lambda \left(\mathfrak{L}^{R,\star}(xe^{\varphi}) - \mathfrak{L}^{R,\star}(x)\right) - (\tilde{r} + \vartheta)\mathfrak{L}^{R,\star}(x) = 0, \quad \text{(A.3.40)}$$

on $x \in (0, \mathfrak{b}_s^R)$ and subject to the boundary conditions

$$\mathfrak{L}^{R,\star}(\mathfrak{b}_s^R) = \mathfrak{b}_s^R - 1 - \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R), \qquad \text{(A.3.41)}$$

$$\partial_x \mathfrak{L}^{R,\star}(\mathfrak{b}_s^R) = 1 - \partial_x \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R), \qquad \text{(A.3.42)}$$

$$\mathfrak{L}^{R,\star}(0) = 0. \qquad \text{(A.3.43)}$$

Therefore, it is sufficient to focus on the non-trivial case, i.e. we assume from now on that $\tilde{r} > \tilde{b}$. Here, we decompose the full domain $[0, \infty)$ into two intervals, $I_1 := [0, \mathfrak{b}_s^R)$ and $I_2 := [\mathfrak{b}_s^R, \infty)$, derive solutions $V_1(\cdot)$ and $V_2(\cdot)$ on these respective domains and combine them to recover $\mathfrak{L}^{R,\star}(\cdot)$ via

$$\mathfrak{L}^{R,\star}(x) = \begin{cases} V_1(x), & x \in I_1, \\ V_2(x), & x \in I_2. \end{cases} \qquad \text{(A.3.44)}$$

We now turn to the derivation of these solutions. First, it is clear that, on I_2, $V_2(x) = x - 1 - \mathfrak{C}_{\mathbf{E}}^{R,\star}(x)$ must hold. Hence, we only need to derive an expression for $V_1(\cdot)$. Here, we start by noting that $\Phi_Y^{(1)}(\theta)$, the Laplace exponent of $(Y_t)_{t \geq 0}$ under $\mathbb{Q}^{(1)}$, is well-defined for all $\theta \in \mathbb{R}$. Furthermore, it can be easily seen that $\theta \mapsto \Phi_Y^{(1)}(\theta)$ is convex and satisfies $\Phi_Y^{(1)}(0) = 0$ and $\lim_{|\theta| \to \infty} \Phi_Y^{(1)}(\theta) = \infty$. Consequently, the equation $\Phi_Y^{(1)}(\theta) = y$ has, for any $y > 0$, two solutions, a positive and a negative root. In the sequel, we denote by $\left(\Phi_Y^{(1)}\right)^{-1,+}(y)$ its positive root and by $\left(\Phi_Y^{(1)}\right)^{-1,-}(y)$ its negative root. Using this notation, one easily shows that, under $\varphi \leq 0$, the general solution of the homogeneous equation (A.3.40) on I_1 takes the form

$$V_1(x) = c_1^+ x^{\gamma_+} + c_1^- x^{\gamma_-}, \qquad \text{(A.3.45)}$$

where $\gamma_+ = \left(\Phi_Y^{(1)}\right)^{-1,+}\left(\tilde{r} + \vartheta\right)$, $\gamma_- = \left(\Phi_Y^{(1)}\right)^{-1,-}\left(\tilde{r} + \vartheta\right)$ and c_1^+, c_1^- are constants to be determined. Therefore, to conclude, we only need to derive c_1^+, c_1^- and $\mathfrak{b}_s^R$ and make use of Conditions (A.3.41)-(A.3.43). First, we note that (A.3.43) implies that $c_1^- \equiv 0$. Additionally, Conditions (A.3.41) and (A.3.42) give the following equations:

$$c_1^+ \left(\mathfrak{b}_s^R\right)^{\gamma_+} = \mathfrak{b}_s^R - 1 - \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R), \qquad \text{(A.3.46)}$$

$$\gamma_+ c_1^+ \left(\mathfrak{b}_s^R\right)^{\gamma_+ - 1} = 1 - \partial_x \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R). \qquad \text{(A.3.47)}$$

The latter system can now be solved to obtain c_1^+ and $\mathfrak{b}_s^R$. First, rewritting (A.3.47) gives that

$$c_1^+ = \frac{\left(\mathfrak{b}_s^R\right)^{1-\gamma_+}}{\gamma_+}\left(1 - \partial_x \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R)\right). \qquad \text{(A.3.48)}$$

Then, inserting this result in (A.3.46) leads to the following non-linear equation in $\mathfrak{b}_s^R$:

$$\mathfrak{b}_s^R = 1 + \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R) + \frac{\mathfrak{b}_s^R}{\gamma_+}\left(1 - \partial_x \mathfrak{C}_{\mathbf{E}}^{R,\star}(\mathfrak{b}_s^R)\right). \qquad \text{(A.3.49)}$$

Therefore, solving the latter equation for $\mathfrak{b}_s^R$ allows us to subsequently derive c_1^+. This finally allows us to recover the tradeability premium $\mathfrak{L}^{R,\star}(\cdot)$ via (A.3.44).

Acknowledgements: I would like to thank Walter Farkas and Giovanni Barone-Adesi for their advice and constructive comments. I also thank Jérôme Detemple, Sander Willems, Alexander Smirnow, Jakub Rojcek, Matthias Feiler and the participants of the Gerzensee Research Days 2018 for their valuable suggestions.

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Chapter 4

Intra-Horizon Expected Shortfall and Risk Structure in Models with Jumps

A version of this paper has been submitted to *Mathematical Finance (Online ISSN: 1467-9965)*.

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Abstract

The present article deals with intra-horizon risk in models with jumps. Our general understanding of intra-horizon risk is along the lines of the approach taken in [BRSW04], [Ro08], [BMK09], [BP10], and [LV20]. In particular, we believe that quantifying market risk by strictly relying on point-in-time measures cannot be deemed a satisfactory approach in general. Instead, we argue that complementing this approach by studying measures of risk that capture the magnitude of losses potentially incurred at any time of a trading horizon is necessary when dealing with (m)any financial position(s). To address this issue, we propose an intra-horizon analogue of the expected shortfall for general profit and loss processes and discuss its key properties. Our intra-horizon expected shortfall is well-defined for (m)any popular class(es) of Lévy processes encountered when modeling market dynamics and constitutes a coherent measure of risk, as introduced in [CDK04]. On the computational side, we provide a simple method to derive the intra-horizon risk inherent to popular Lévy dynamics. Our general technique relies on results for maturity-randomized first-passage probabilities and allows for a derivation of diffusion and single jump risk contributions. These theoretical results are complemented with an empirical analysis, where popular Lévy dynamics are calibrated to S&P 500 index data and an analysis of the resulting intra-horizon risk is presented.

Keywords: Intra-Horizon Risk, Value at Risk, Expected Shortfall, Lévy Processes, Hyper-Exponential Distribution, Risk Decomposition.

MSC (2010) Classification: 91-08, 91B25, 91B30, 91B70, 91B82, 91G60, 91G80.

JEL Classification: C32, C63, G01, G51.

4.1 Introduction

For the past 20 years, the (point-in-time) value at risk, defined as quantile of the profit and loss distribution at the end of a predefined trading horizon, has been one of the most widely used measure of market risk for regulatory capital allocation (cf. [BCBS06], [BCBS19]). Despite its popularity, this risk measure has several major drawbacks that are all known to academics since many years. Firstly, it is mainly concerned with the probability of a loss and not with the actual loss size itself. In particular, when relying on the (point-in-time) value at risk, the distribution of the losses that exceed the quantile of interest is not taken into account. Secondly, it does not satisfy the subadditivity property for monetary risk measures (cf. [ADEH99], [RT02], [AT02], [EPRWB14]) and therefore does not constitute a coherent measure of risk in the sense of [ADEH99]. Lastly, as a measure of market risk, the (point-in-time) value at risk does not capture the full magnitude of losses that may be potentially incurred at any time of a trading horizon (cf.[BRSW04], [Ro08], [BMK09]).

To address some of these issues, two streams have emerged in the academic literature. While certain authors (cf. [ADEH99], [RT02], [AT02]) introduced the (point-in-time) expected shortfall, a coherent risk measure that additionally depends on the tail of the underlying profit and loss distribution at the end of a predefined trading horizon, other authors (cf. [BRSW04], [Ro08], [BMK09], [BP10], [LV20]) developed a new path-dependent market risk measure, the intra-horizon value at risk. In light of the recent admission of the (point-in-time) expected shortfall in the new market risk framework of the Basel Accords and of the constant demand therein for sufficient conservatism in the risk estimates (cf. [BCBS19]), we believe that it is high time to reunify these two branches of the academic literature and to combine their advantages to propose a novel, coherent and path-dependent market risk measure that depends on all extremes in a trading horizon. This is the content of the present article that extensively discusses an intra-horizon version of the (point-in-time) expected shortfall.

Our paper's contribution is manifold and has both theoretical and practical relevance. On the theoretical side, we first generalize the current intra-horizon risk quantification approach of the literature (cf. [BRSW04], [Ro08], [BMK09], [BP10], [LV20]) and propose an intra-horizon analogue of the expected shortfall for general profit and loss processes. The resulting risk measure has several desirable properties and constitutes a coherent measure of risk in the sense of [CDK04]. Additionally, we show that our general ansatz is linked to simple (and maturity-randomized) first-passage probabilities of the underlying profit and loss process and subsequently use these relations to prove that the intra-horizon expected shortfall is well-defined for (m)any popular Lévy dynamics encountered in financial modeling. Secondly, we introduce diffusion and jump contributions to first-passage occurrences under Lévy models and present characterizations of diffusion and jump contributions to simple and maturity-randomized first-passage probabilities. These characterizations are then used in the following way: First, diffusion and jump risk contributions to the intra-horizon expected shortfall are inferred. Second, (semi-)analytical results for diffusion and jump contributions to maturity-randomized first-passage probabilities are derived under the class of hyper-exponential jump-diffusion processes by relying on option pricing methods (cf. among others [Ca09], [CCW09], [CK12], [CYY13], [HM13], [AR16], [LV17], [LV20]).

On the practical side, we introduce a simple and efficient ansatz to compute the intra-horizon risk inherent to popular Lévy dynamics. Our general approach consists in combining hyper-exponential jump-diffusion approximations to (pure jump) Lévy processes having completely monotone jumps with our (semi-)analytical results in this class of processes to recover approximate, though arbitrarily close intra-horizon risk results for the original process. In doing so, we rely on similar ideas to the ones introduced in [AMP07], [JP10], and [LV20] and subsequently use a mix of our results for maturity-randomized first-passage probabilities and a Laplace inversion algorithm (cf. [Co07]) to arrive at intra-horizon expected shortfall results.[1] Lastly, as an application of the techniques developed in this paper, we calibrate S&P 500 index data to popular Lévy dynamics and investigate the intra-horizon risk inherent to a long position in the S&P 500 index from January 1995 to April 2019. Our empirical findings reveal that even for high loss quantiles (i.e. low α) the intra-horizon value at risk and the intra-horizon expected shortfall add conservatism to their point-in-time estimates. Additionally, they suggest that these risk measures have a very similar structure across jumps/jump clusters and that already a high contribution of their risk is due to only few, large – in terms of the absolute jump size – jump clusters.

The remaining of this paper is structured as follows. In Section 4.2, we introduce our general intra-horizon risk quantification approach as well as the notation used in the rest of the paper. This section also links intra-horizon risk to first-passage probabilities of the underlying profit and loss process. Section 4.3 deals with intra-horizon risk in models with jumps and is divided into two parts. Firstly, our intra-horizon risk quantification approach is further developed under the assumption of Lévy dynamics and characterizations of simple (and maturity-randomized) first-passage probabilities are discussed. These characterizations are secondly used to derive (semi-)analytical results for maturity-randomized first-passage probabilities under the class of hyper-exponential jump-diffusion processes. Section 4.4 reviews hyper-exponential jump-diffusion approximations to pure jump Lévy processes having a completely monotone jump density as well as few adaptions. All the theoretical results of Sections 4.2-4.4 are lastly combined in Section 4.5, where popular Lévy dynamics are calibrated to S&P 500 index data and the intra-horizon risk inherent to a long position in the S&P 500 index is analyzed from January 1995 to April 2019. The paper concludes with Section 4.6. All proofs and complementary results are presented in the Appendices (Appendix A, B and C; Section 4.7).

4.2 Fundamental Concepts of Intra-Horizon Risk Quantification

We start by discussing the problem of evaluating the intra-horizon risk inherent to a financial position. To this end, we fix a time horizon $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, whose filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. We let $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ be an $\mathbf{F}$-adapted real-valued stochastic process and interpret its realizations as possible discounted profit and loss realizations of a given financial position over the valuation horizon $[0, T]$. Here, we do not necessarily require the process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ to start at $\mathcal{P}\&\mathcal{L}_0 = 0$ but allow instead for more flexibility in the choice of its initial value, i.e. we let $\mathcal{P}\&\mathcal{L}_0 = z$ for general $z \in \mathbb{R}$. This generalization proves useful, when rolling profits/losses of financial positions over multiple valuation periods. In this case, $\mathcal{P}\&\mathcal{L}_0$ represents the profit/loss accumulated from the establishment of the position until the start of the valuation horizon under consideration.

4.2.1 Intra-Horizon Value at Risk

Our understanding of intra-horizon risk is in line with the ideas presented in [BRSW04], [BP10] and [LV20]. As in these papers, our focus is on market risk, i.e. we only deal with risk that arises out of movements in the market price of financial assets and fully abstract from other risk types, such as e.g. counterparty risk. Additionally, we believe that quantifying market risk by strictly relying on point-in-time measures cannot be deemed a satisfactory approach in general. Instead, complementing this approach by studying measures of risk that capture the magnitude of losses potentially incurred at any time of a trading horizon is necessary for many asset types. This motivates the consideration of the minimum (discounted) profit and loss process,

[1] We choose the Gaver-Stehfest algorithm that has the particularity to allow for an inversion of the transform on the real line and that has been successfully used by several authors in the option pricing literature (cf. [KW03], [Ki10], [HM13], [LV17]).

$(I_t^{\mathcal{P}\&\mathcal{L}})_{t\in[0,T]}$, that is defined via

$$I_t^{\mathcal{P}\&\mathcal{L}} := \inf_{0\le u\le t} \mathcal{P}\&\mathcal{L}_u, \qquad t \in [0,T]. \tag{4.2.1}$$

Under this notation, the following definition of the intra-horizon value at risk was presented in [BP10].

Definition 4.1 (Intra-Horizon Value at Risk). *Let $T > 0$ and $\alpha \in (0,1)$ be fixed. The level-α intra-horizon value at risk associated with the (discounted) profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ over the time interval $[0,T]$, $iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})$, is defined as*

$$iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L}) := V@R_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big), \tag{4.2.2}$$

where, for a random variable Y, $V@R_\alpha(Y)$ denotes the (point-in-time) value at risk to the level α and is defined as

$$V@R_\alpha(Y) := -q_\alpha(Y). \tag{4.2.3}$$

Here, $q_\alpha(Y)$ denotes the upper α-quantile of Y, which is obtained via

$$q_\alpha(Y) := \sup\{y \in \mathbb{R} :\ \mathbb{P}\,(Y \le y) \le \alpha\}. \tag{4.2.4}$$

Specifying the intra-horizon value at risk in the above sense is in particular linked to the theory of ruin that has received a lot of attention in insurance mathematics. Indeed, the above definition can be clearly re-expressed in terms of first-passage probabilities, as

$$iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L}) := -\sup\Big\{\ell \in \mathbb{R} :\ \mathbb{P}_z\big(\tau_\ell^{\mathcal{P}\&\mathcal{L},-} \le T\big) \le \alpha\Big\}, \tag{4.2.5}$$

where we denote by $\mathbb{P}_z$ the probability measure under which the process $(\mathcal{P}\&\mathcal{L})_{t\in[0,T]}$ starts at $z \in \mathbb{R}$ and we use, for $\ell \in \mathbb{R}$ and a given stochastic process $(Y_t)_{t\in[0,T]}$, the notation

$$\tau_\ell^{Y,\pm} := \inf\{t \ge 0 :\ \pm Y_t \ge \pm\ell\}, \qquad \text{with} \qquad \inf\emptyset = \infty. \tag{4.2.6}$$

This representation will prove useful, as it will allow us to combine properties of first-passage probabilities to subsequently recover intra-horizon value-at-risk results via standard numerical methods.

4.2.2 Intra-Horizon Expected Shortfall

Although the intra-horizon value at risk already accounts for intra-horizon features, it suffers from two major drawbacks. Firstly, it is mainly concerned with the probability of a loss and not with the actual loss size itself. In particular, when relying on the intra-horizon value at risk, the distribution of the losses that exceed the quantile of interest is not taken into account. Secondly, it is not subadditive (cf. [BP10]) and therefore does not constitute a coherent measure of risk in the sense of [CDK04].[2] This is due to the fact that the intra-horizon value at risk consists of an adaption of the point-in-time value at risk, which in its turn is known to have the same deficiencies. To address these major shortcomings, we propose a (market) risk measure that defines an intra-horizon analogue of the expected shortfall. This is the content of the next definition, where we use the notation $\mathbb{E}_z^{\mathbb{P}}[\cdot]$ to indicate expectation under the measure $\mathbb{P}_z$.

[2] Like its point-in-time counterpart, the intra-horizon value at risk may become superadditive for certain profit and loss processes and therefore defies the notion of diversification.

Definition 4.2 (Intra-Horizon Expected Shortfall). *Let $T > 0$ and $\alpha \in (0,1)$ be fixed and assume that $\mathbb{E}_z^{\mathbb{P}}\left[|I_T^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$. Then, the level-$\alpha$ intra-horizon expected shortfall associated with the (discounted) profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ over the time horizon $[0,T]$, $iES_{\alpha,T}(\mathcal{P}\&\mathcal{L})$, is defined as*

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) := \frac{1}{\alpha}\int_0^{\alpha} iV@R_{\gamma,T}(\mathcal{P}\&\mathcal{L})\, d\gamma. \tag{4.2.7}$$

It is not hard to see that the above specification of the intra-horizon expected shortfall overcomes both major shortcomings of the intra-horizon value at risk. Indeed, a brief look at equation (4.2.7) reveals that our intra-horizon expected shortfall defines a coherent measure of risk in the sense of [CDK04] that additionally depends on the distributional properties in the tail of the underlying profit and loss process. In fact, cash-invariance, monotonicity and positive homogeneity of (4.2.7) directly follow from the corresponding properties of the point-in-time expected shortfall via the identity

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) = \frac{1}{\alpha}\int_0^{\alpha} V@R_{\gamma}(I_T^{\mathcal{P}\&\mathcal{L}})\, d\gamma =: ES_{\alpha}\big(I_T^{\mathcal{P}\&\mathcal{L}}\big). \tag{4.2.8}$$

Additionally, subadditivity is obtained by relying on the monotonicity and subadditivity properties of the point-in-time expected shortfall and the fact that for two profit and loss processes $\big(\mathcal{P}\&\mathcal{L}_t^1\big)_{t\in[0,T]}$ and $\big(\mathcal{P}\&\mathcal{L}_t^2\big)_{t\in[0,T]}$ the following identity holds

$$I_T^{\mathcal{P}\&\mathcal{L}^1+\mathcal{P}\&\mathcal{L}^2} \geq I_T^{\mathcal{P}\&\mathcal{L}^1} + I_T^{\mathcal{P}\&\mathcal{L}^2}. \tag{4.2.9}$$

Then,

$$\begin{aligned} iES_{\alpha,T}\big(\mathcal{P}\&\mathcal{L}^1 + \mathcal{P}\&\mathcal{L}^2\big) &= ES_{\alpha}\big(I_T^{\mathcal{P}\&\mathcal{L}^1+\mathcal{P}\&\mathcal{L}^2}\big) \leq ES_{\alpha}\big(I_T^{\mathcal{P}\&\mathcal{L}^1} + I_T^{\mathcal{P}\&\mathcal{L}^2}\big) \\ &\leq ES_{\alpha}\big(I_T^{\mathcal{P}\&\mathcal{L}^1}\big) + ES_{\alpha}\big(I_T^{\mathcal{P}\&\mathcal{L}^2}\big) = iES_{\alpha,T}\big(\mathcal{P}\&\mathcal{L}^1\big) + iES_{\alpha,T}\big(\mathcal{P}\&\mathcal{L}^2\big), \end{aligned} \tag{4.2.10}$$

which is the subadditivity property.

The next proposition indicates the link between the intra-horizon expected shortfall and the theory of ruin. In particular, it shows that, whenever well-defined, the difference between intra-horizon expected shortfall and intra-horizon value at risk can be computed for any given profit and loss process based on first-passage probabilities. The proof is provided in Appendix A (cf. Section 4.7.1).

Proposition 4.1. *Let $T > 0$ and $\alpha \in (0,1)$ be fixed and assume that $\mathbb{E}_z^{\mathbb{P}}\left[|I_T^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$. Then, the level-$\alpha$ intra-horizon expected shortfall can be re-expressed in the form*

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) = \frac{1}{\alpha}\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} \mathbb{P}_z\big(\tau_{\ell}^{\mathcal{P}\&\mathcal{L},-} \leq T\big)\, d\ell + iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L}). \tag{4.2.11}$$

Remark 4.1.

i) Combining Proposition 4.1 with Representation (4.2.5) leads to an important implication – both the intra-horizon value at risk and the intra-horizon expected shortfall can be fully characterized based on first-passage probabilities. Therefore, we will study, for $T > 0$, $z \in \mathbb{R}$ and $\ell \leq 0$ the function

$$u(\mathcal{T}, z; \ell) := \mathbb{P}_z\big(\tau_{\ell}^{\mathcal{P}\&\mathcal{L},-} \leq \mathcal{T}\big) = \mathbb{E}_z^{\mathbb{P}}\left[\mathbb{1}_{\{\tau_{\ell}^{\mathcal{P}\&\mathcal{L},-} \leq \mathcal{T}\}}\right] \quad \text{with} \quad \mathcal{T} \in [0,T], \tag{4.2.12}$$

and subsequently recover intra-horizon measures of risk via numerical techniques. Here, $\mathcal{T} \in [0,T]$ refers to the remaining time to maturity and is linked to any pair of times (t,T) satisfying $0 \leq t \leq T$ via $\mathcal{T} = T - t$.

ii) Besides providing an important link to first-passage probabilities, Equation (4.2.11) formalizes the intuitive property that the intra-horizon expected shortfall always exceeds the intra-horizon value at risk.

♦

4.3 Intra-Horizon Risk and Models with Jumps

We next turn to a discussion of intra-horizon risk under infinitely divisible distributions, i.e. we fix an $\mathbf{F}$-Lévy process $(X_t)_{t\in[0,T]}$ and consider two different scenarios:

– *Scenario 1*, where the dynamics of the (discounted) profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ are directly described by $(X_t)_{t\in[0,T]}$, i.e. where

$$\mathcal{P}\&\mathcal{L}_t = X_t, \qquad t \in [0,T].$$

– *Scenario 2*, where the (discounted) profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ reflects the intrinsic value of a long (+) or short (−) position in an asset of ordinary exponential Lévy type, i.e. where we have that

$$\mathcal{P}\&\mathcal{L}_t = \pm\left(z_1 e^{X_t} - z_2\right), \quad t \in [0,T], \qquad \text{with } z_1, z_2 \in \mathbb{R}_0^+.$$

4.3.1 Lévy Processes and Notation

We recall that a Lévy process $(X_t)_{t\geq 0}$ on a (filtered) probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}^X)$ is a càdlàg (right-continuous with left limits) process having independent and stationary increments and Lévy-exponent $\Psi_X(\cdot)$ defined, for $\theta \in \mathbb{R}$, in terms of its characteristic triplet (b_X, σ_X^2, Π_X) via

$$\Psi_X(\theta) := -\log\left(\mathbb{E}^{\mathbb{P}^X}\left[e^{i\theta X_1}\right]\right) = -ib_X\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\leq 1\}}\right)\Pi_X(dy), \tag{4.3.1}$$

where $\mathbb{E}^{\mathbb{P}^X}[\cdot]$ indicates expectation under the measure $\mathbb{P}^X$. Well known results in the theory of Lévy processes (cf. [Sa99], [Ap09]) allow to decompose $(X_t)_{t\geq 0}$ in terms of its jump and diffusion parts as

$$X_t = b_X t + \sigma_X W_t + \int_{\mathbb{R}} y\, \bar{N}_X(t, dy), \qquad t \geq 0, \tag{4.3.2}$$

where $(W_t)_{t\geq 0}$ denotes an $\mathbf{F}$-Brownian motion and N_X refers to an independent Poisson random measure on $[0,\infty) \times \mathbb{R} \setminus \{0\}$ that has intensity measure given by Π_X. Here, we use for any $t \geq 0$ and Borel set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ the following notation:

$$\begin{aligned} N_X(t, A) &:= N_X((0,t] \times A), \\ \tilde{N}_X(dt, dy) &:= N_X(dt, dy) - \Pi_X(dy)dt, \\ \bar{N}_X(dt, dy) &:= \begin{cases} \tilde{N}_X(dt, dy), & \text{if } |y| \leq 1, \\ N_X(dt, dy), & \text{if } |y| > 1. \end{cases} \end{aligned}$$

Additionally, we define the Laplace exponent of $(X_t)_{t\geq 0}$, for any $\theta \in \mathbb{R}$ satisfying $\mathbb{E}^{\mathbb{P}^X}\left[e^{\theta X_1}\right] < \infty$, via the following identity

$$\Phi_X(\theta) := -\Psi_X(-i\theta) = b_X\theta + \frac{1}{2}\sigma_X^2\theta^2 - \int_{\mathbb{R}} \left(1 - e^{\theta y} + \theta y \mathbb{1}_{\{|y|\leq 1\}}\right)\Pi_X(dy), \tag{4.3.3}$$

and recall that $(X_t)_{t\geq 0}$ has the (strong) Markov property. Therefore, its infinitesimal generator is a partial integro-differential operator given, for sufficiently smooth $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, by

$$\begin{aligned}\mathcal{A}_X V(\mathcal{T}, x) &:= \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathbb{P}^X}\left[V(\mathcal{T}, X_t)\right] - V(\mathcal{T}, x)}{t} \\ &= \frac{1}{2}\sigma_X^2\partial_x^2 V(\mathcal{T}, x) + b_X\partial_x V(\mathcal{T}, x) + \int_{\mathbb{R}} \left[V(\mathcal{T}, x+y) - V(\mathcal{T}, x) - y\mathbb{1}_{\{|y|\leq 1\}}\partial_x V(\mathcal{T}, x)\right]\Pi_X(dy),\end{aligned} \tag{4.3.4}$$

where the expectation is taken under the measure $\mathbb{P}_x^X$ having initial distribution $X_0 = x$. We will extensively make use of these notations in the upcoming sections.

4.3.2 First-Passage Decomposition and Intra-Horizon Risk under the Lévy Framework

Dealing with first-passage events in any of *Scenario 1* and *Scenario 2* clearly reduces to the consideration of corresponding events for simple Lévy processes. This follows from the properties of the exponential function as well as from the fact that, under both scenarios, the process $(X_t)_{t\in[0,T]}$ is the only source of uncertainty. Consequently, we only need to study, for $x \in \mathbb{R}$ and $L \in \mathbb{R}$, the first-passage probabilities defined by

$$u_X^{\pm}(\mathcal{T}, x; L) := \mathbb{P}_x^X\left(\tau_L^{X,\pm} \leq \mathcal{T}\right) = \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_L^{X,\pm}\leq \mathcal{T}\}}\right] \qquad \text{for} \qquad \mathcal{T} \in [0, T], \tag{4.3.5}$$

and note that (4.2.12) and (4.3.5) are related with each other by means of the following identity

$$u(\mathcal{T}, z; \ell) = \begin{cases} u_X^-(\mathcal{T}, z; \ell), & \text{for } \textit{Scenario 1}, \\ u_X^-\big(\mathcal{T}, \log(z_2 + z); \log(z_2 + \ell)\big), & \text{for a long position in } \textit{Scenario 2}, \\ u_X^+\big(\mathcal{T}, \log(z_2 - z); \log(z_2 - \ell)\big), & \text{for a short position in } \textit{Scenario 2}. \end{cases} \tag{4.3.6}$$

At this point, we should note that not all parameters $z, \ell \in \mathbb{R}$ lead to sensible results when dealing with *Scenario 2*. Therefore, under this scenario, the above formula should be always understood on the respective ranges, i.e. we set $\log(0) := -\infty$ and only consider the following values for z, ℓ:

i) $z, \ell \in [-z_2, \infty)$ for a long position,

ii) $z, \ell \in (-\infty, z_2]$ for a short position.

The next lemma proves useful when dealing with intra-horizon risk in models with jumps and provides simple conditions on the Laplace exponent of the underlying Lévy process for which intra-horizon expected shortfall measures are well-defined. A proof is provided in Appendix A (cf. Section 4.7.1).

Lemma 4.1. *Let $(X_t)_{t\geq 0}$ be a Lévy process and assume that the following condition is satisfied:*

$$\exists \theta^\star > 1: \quad \mathbb{E}_0^{\mathbb{P}^X}\left[e^{\theta^\star |X_1|}\right] < \infty. \tag{4.3.7}$$

Then, there exists a constant $c > 1$ such that for any $x \in \mathbb{R}$ and $\mathcal{T} > 0$ we have

$$\lim_{\ell \uparrow \infty} e^{c\cdot \ell}\, \mathbb{P}_x^X\left(\tau_\ell^{X,+} \leq \mathcal{T}\right) = 0 \quad \text{and} \quad \lim_{\ell \downarrow -\infty} e^{-c\cdot \ell}\, \mathbb{P}_x^X\left(\tau_\ell^{X,-} \leq \mathcal{T}\right) = 0. \tag{4.3.8}$$

In particular, these convergence results ensure that $\mathbb{E}_z^{\mathbb{P}}\left[|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$ holds for any $\mathcal{T} > 0$, or, equivalently, that the intra-horizon expected shortfall is well-defined under any of Scenario 1 and Scenario 2.

Although Condition (4.3.7) slightly restricts the applicability of Lemma 4.1, many popular classes of Lévy processes encountered in financial applications satisfy this property, at least on a range of parameters that is suitable for market modeling purposes. Important examples include hyper-exponential jump-diffusion models (cf. [Ko02], [Ca09]), Variance Gamma (VG) processes (cf. [MS90], [MCC98]), the Carr-Geman-Madan-Yor (CGMY) model (cf. [CGMY02]) as well as Normal Inverse Gaussian (NIG) processes (cf. [BN97]). We will deal with the intra-horizon risk inherent to some of these models in our numerical analysis of Section 4.5.

Remark 4.2.

i) A closer look at the proof of Lemma 4.1 reveals that under each of the scenarios under consideration, only one of the conditions

$$\lim_{\ell \uparrow \infty} e^{c\cdot \ell}\, \mathbb{P}_x^X\left(\tau_\ell^{X,+} \leq \mathcal{T}\right) = 0, \quad \text{or} \quad \lim_{\ell \downarrow -\infty} e^{-c\cdot \ell}\, \mathbb{P}_x^X\left(\tau_\ell^{X,-} \leq \mathcal{T}\right) = 0,$$

is sufficient to ensure that $\mathbb{E}_z^{\mathbb{P}}\left[|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$ holds for $\mathcal{T} > 0$. Additionally, the proof of Lemma 4.1 indicates that these properties are consequences of the corresponding one-sided conditions

$$\exists \theta^\star > 1: \quad \mathbb{E}_0^{\mathbb{P}^X}\left[e^{\theta^\star X_1}\right] < \infty, \quad \text{and} \quad \exists \theta_\star < -1: \mathbb{E}_0^{\mathbb{P}^X}\left[e^{\theta_\star X_1}\right] < \infty, \tag{4.3.9}$$

respectively, which therefore even weaken the requirements on the dynamics of the process $(X_t)_{t\geq 0}$.

ii) When considering a long position in *Scenario 2*, well-definedness of the intra-horizon expected shortfall is immediate. In this case, the condition $\mathbb{E}_z^{\mathbb{P}}\left[|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$ directly follows, for any $\mathcal{T} > 0$, from the fact that $I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \geq -z_2$, i.e. that the maximal possible losses do not exceed the value z_2. We will dedicate our numerical analysis in Section 4.5 to exactly this scenario and investigate the intra-horizon risk inherent to a long position in the S&P 500 index using weekly data ranging from January 1990 to April 2019.

♦

4.3.2.1 First-Passage Decomposition and PIDEs

For risk management purposes, it may be of great importance to further understand the structure of risk. In particular, one may want to know how often certain shortfall barriers are already exceeded at the time they are breached. When dealing with market risk, this roughly reduces to the question of whether first-passage occurrences are triggered by diffusion or by jumps. This question can be further investigated under the present Lévy framework and a first-passage decomposition can be obtained.[3] This is discussed next. Here, we start from a similar approach to the one taken in [CCW09] and define, for any $\ell \in \mathbb{R}$, the following events

$$\mathcal{E}_0^{\pm} := \left\{ X_{\tau_\ell^{X,\pm}} = \ell \right\} \qquad \text{and} \qquad \mathcal{E}_{\mathcal{J}}^{\pm} := \left\{ X_{\tau_\ell^{X,\pm}} \neq \ell \right\}, \tag{4.3.10}$$

i.e. we essentially decompose the first-passage times in events that are either triggered by the diffusion part of the process $(X_t)_{t\in[0,T]}$ or by jumps.[4] Clearly, the first-passage events $\mathcal{E}_0^+$ and $\mathcal{E}_{\mathcal{J}}^+$ are disjoint and the same additionally holds for $\mathcal{E}_0^-$ and $\mathcal{E}_{\mathcal{J}}^-$. Hence, this allows us to obtain, for $\mathcal{T} \in [0,T]$, $x \in \mathbb{R}$ and $\ell \in \mathbb{R}$, a decomposition of the first-passage probabilities as

$$u_X^{\pm}(\mathcal{T}, x; \ell) = u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) + u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell), \tag{4.3.11}$$

where $u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell)$ and $u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell)$ refer to the functions defined by

$$u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) := \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\left\{\tau_\ell^{X,\pm} \le \mathcal{T}\right\} \cap \mathcal{E}_0^{\pm}} \right] \qquad \text{and} \qquad u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell) := \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\left\{\tau_\ell^{X,\pm} \le \mathcal{T}\right\} \cap \mathcal{E}_{\mathcal{J}}^{\pm}} \right]. \tag{4.3.12}$$

We now turn to an analysis of these first-passage probabilities and first aim to obtain a PIDE characterization of the functions $u_X^{\mathcal{E}_0^{\pm}}(\cdot)$ and $u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\cdot)$. To this end, we define, for $\ell \in \mathbb{R}$, the following domains

$$\mathcal{H}_\ell^+ := (\ell, \infty) \qquad \text{and} \qquad \mathcal{H}_\ell^- := (-\infty, \ell) \tag{4.3.13}$$

and denote by $\overline{\mathcal{H}^{\pm}}$ the closure of these sets in $\mathbb{R}$. Under this notation, the next proposition is obtained by relying on (strong) Markovian arguments. A proof is provided in Appendix B (cf. Section 4.7.2).

Proposition 4.2. *For any level $\ell \in \mathbb{R}$, the first-passage probability contributed by the diffusion part, $u_X^{\mathcal{E}_0^{\pm}}(\cdot)$, satisfies the following Cauchy problem:*

$$-\partial_{\mathcal{T}} u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) + \mathcal{A}_X u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) = 0, \qquad \text{on } (\mathcal{T}, x) \in (0,T] \times \left(\mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}} \right), \tag{4.3.14}$$

$$u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) = 1, \qquad \text{on } (\mathcal{T}, x) \in [0,T] \times \{\ell\}, \tag{4.3.15}$$

$$u_X^{\mathcal{E}_0^{\pm}}(\mathcal{T}, x; \ell) = 0, \qquad \text{on } (\mathcal{T}, x) \in [0,T] \times \mathcal{H}_\ell^{\pm}, \tag{4.3.16}$$

$$u_X^{\mathcal{E}_0^{\pm}}(0, x; \ell) = 0, \qquad \text{on } x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}}. \tag{4.3.17}$$

Similarly, the first-passage probability contributed by jumps, $u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\cdot)$, solves, for any $\ell \in \mathbb{R}$, the Cauchy problem

$$-\partial_{\mathcal{T}} u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell) + \mathcal{A}_X u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell) = 0, \qquad \text{on } (\mathcal{T}, x) \in (0,T] \times \left(\mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}} \right), \tag{4.3.18}$$

$$u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell) = 0, \qquad \text{on } (\mathcal{T}, x) \in [0,T] \times \{\ell\}, \tag{4.3.19}$$

$$u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\mathcal{T}, x; \ell) = 1, \qquad \text{on } (\mathcal{T}, x) \in [0,T] \times \mathcal{H}_\ell^{\pm}, \tag{4.3.20}$$

$$u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(0, x; \ell) = 0, \qquad \text{on } x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}}. \tag{4.3.21}$$

[3] As we shall see in a moment, the same approach can be adopted for any strong Markov process that is quasi-left-continuous.

[4] We emphasize that this interpretation may not be (fully) correct in cases where $\tau_\ell^{X,\pm} = \infty$, however, that these cases are subsequently excluded from our analysis, as seen in (4.3.12). Additionally, we note that for general jump dynamics parts of the event $\mathcal{E}_0^{\pm}$ could be due to jumps and that it may be therefore more appropriate to speak of a first-passage decomposition in events with and without overshoot. Nevertheless, since market models usually assume continuous jump distributions, i.e. an intensity measure Π_X of the form $\Pi_X(dy) = \pi_X(y)dy$, with appropriate jump density $\pi_X(\cdot)$, this situation will not occur. This justifies the use of our initial terminology.

Combining the characterization in Proposition 4.2 with (standard) numerical techniques already allows for a numerical treatment of the functions $u_X^{\mathcal{E}_0^{\pm}}(\cdot)$ and $u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\cdot)$. Furthermore, the proof of Proposition 4.2 reveals that our derivations are not restricted to the Lévy framework. Indeed, while the decomposition in (4.3.11) is a simple consequence of the disjointness of the sets defined in (4.3.10), the proof of Proposition 4.2 combines (strong) Markovian arguments with the quasi-left-continuity of the process $(X_t)_{t\in[0,T]}$. As a consequence, relying on this approach is always possible when dealing with processes that satisfy these two properties and a disentanglement of diffusion and jump contributions can be then obtained via the exact same techniques.

Remark 4.3.

In general, the above techniques can be applied to subsequently recover intra-horizon risk measures as well as corresponding risk contributions. However, even when dealing with the intra-horizon value at risk to a single level $\alpha \in (0,1)$ numerous iterations of the numerical scheme are needed and the computational costs quickly become high. Therefore, relying on these techniques for expected shortfall measures does not seem to be the best approach. Instead, distributional properties inherent to certain distributions sometimes allow to simplify the problem by switching to maturity-randomization. This holds for instance true when dealing with hyper-exponential jump-diffusion processes that have the particularity to allow for arbitrarily close approximations of Lévy processes with completely monotone jumps (cf. [JP10], [CK11], [HK16]). A discussion of this approach as well as of approximations of Lévy densities via hyper-exponential jump densities is provided in the upcoming sections.

♦

4.3.2.2 Maturity-Randomization and OIDEs

We next deal with maturity-randomized first-passage probabilities. To this end, we start by defining for any function $g : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\int_0^\infty e^{-\vartheta t} |g(t)|\, dt < \infty, \qquad \forall \vartheta > 0, \tag{4.3.22}$$

the Laplace-Carson transform $\mathcal{LC}(g)(\cdot)$ via

$$\mathcal{LC}(g)(\vartheta) := \int_0^\infty \vartheta e^{-\vartheta t}\, g(t)\, dt, \tag{4.3.23}$$

and note that this transform has several desirable properties. First, the Laplace-Carson transform merely corresponds to a scaled Laplace transform, for which extensive inversion techniques exist (cf. [Co07]). Additionally, as we shall see in a moment, applying the Laplace-Carson transform in the context of mathematical finance allows to randomize the maturity of financial contracts, i.e. to switch from objects with deterministic maturity to corresponding objects with stochastic maturity. This last property offers a range of alternative ways to tackle problems related to the valuation of financial positions and has therefore led to a wide adoption of the Laplace-Carson transform in the option pricing literature, with [Ca98] being one of the seminal articles in this context.

Having computed the transform (either numerically or analytically), the original function $g(\cdot)$ can be recovered from $\mathcal{LC}(g)(\cdot)$ using an inversion algorithm. One possible choice is the Gaver-Stehfest algorithm that has the particularity to allow for an inversion of the transform on the real line and that has been successfully used by several authors for option pricing (cf. [KW03], [Ki10], [HM13], [LV17], [CV18]). We will also rely on this algorithm, i.e. we set

$$g_N(t) := \sum_{k=1}^{2N} \zeta_{k,N}\, \mathcal{LC}\big(g\big)\left(\frac{k\log(2)}{t}\right), \qquad N \in \mathbb{N},\ t > 0, \tag{4.3.24}$$

where the coefficients are given by

$$\zeta_{k,N} := \frac{(-1)^{N+k}}{k} \sum_{j=\lfloor (k+1)/2 \rfloor}^{\min\{k,N\}} \frac{j^{N+1}}{N!} \binom{N}{j}\binom{2j}{j}\binom{j}{k-j}, \qquad N \in \mathbb{N},\ 1 \leq k \leq 2N, \tag{4.3.25}$$

with $\lfloor a \rfloor := \sup\{z \in \mathbb{Z} :\ z \leq a\}$, and will recover the original function $g(\cdot)$ by means of the following relation

$$\lim_{N\to\infty} g_N(t) = g(t). \tag{4.3.26}$$

More details on the Gaver-Stehfest algorithm as well as formal proofs of the convergence result (4.3.26) for "sufficiently well-behaved functions" are provided in [Ku13] and references therein.

We now turn to a discussion of Laplace-Carson transformed first-passage probabilities. First, we note that the boundedness of the functions $u_X^{\pm}(\cdot)$ and $u_X^{\mathcal{E}}(\cdot)$ for $\mathcal{E} \in \{\mathcal{E}_0^{\pm}, \mathcal{E}_{\mathcal{J}}^{\pm}\}$ ensures that these first-passage probabilities satisfy Condition (4.3.22) and so that the resulting Laplace-Carson transform is well-defined. Additionally, one easily sees that the first-passage decompositions obtained in (4.3.11) are preserved under the Laplace-Carson operator, i.e. we have for any $\vartheta > 0$, $x \in \mathbb{R}$ and $\ell \in \mathbb{R}$ that

$$\mathcal{LC}\big(u_X^{\pm}\big)(\vartheta, x; \ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta, x; \ell) + \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta, x; \ell). \tag{4.3.27}$$

This property is particularly interesting since it implies that switching back and forth between the original first-passage probabilities and their corresponding Laplace-Carson transforms does not alter the structure of risk across the diffusion and jump parts and therefore allows us to fully concentrate on one or the other. Finally, any of the Laplace-Carson transformed first-passage probabilities can be interpreted as the probability of a respective first-passage occurring before an independent exponentially distributed random time of intensity $\vartheta > 0$, $\mathcal{T}_\vartheta$, has expired or equivalently before the first jump time of an independent Poisson process $(N_t)_{t\geq 0}$ having intensity $\vartheta > 0$ has happened. This is easily seen from the following identities

$$\mathcal{LC}\big(u_X^{\pm}\big)(\vartheta, x; \ell) = \int_0^\infty \vartheta e^{-\vartheta t}\, u_X^{\pm}(t, x; \ell)\, dt = \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,\pm} \leq \mathcal{T}_\vartheta\}} \,\middle|\, \mathcal{T}_\vartheta\right]\right] = \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,\pm} \leq \mathcal{T}_\vartheta\}}\right], \tag{4.3.28}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}}\big)(\vartheta, x; \ell) = \int_0^\infty \vartheta e^{-\vartheta t}\, u_X^{\mathcal{E}}(t, x; \ell)\, dt = \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,\pm} \leq \mathcal{T}_\vartheta\} \cap \mathcal{E}} \,\middle|\, \mathcal{T}_\vartheta\right]\right] = \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,\pm} \leq \mathcal{T}_\vartheta\} \cap \mathcal{E}}\right], \tag{4.3.29}$$

where $\mathcal{E} \in \big\{\mathcal{E}_0^{\pm}, \mathcal{E}_{\mathcal{J}}^{\pm}\big\}$. Consequently, any application of the Laplace-Carson operator transforms (in this context) first-passage probabilities into corresponding maturity-randomized quantities and combining these properties with arguments similarly used in the proof of Proposition 4.2 allows us to obtain an OIDE characterization of the maturity-randomized first-passage probabilities contributed by the diffusion part, $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$, and by jumps, $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$. This is the content of the next proposition, whose proof is provided in Appendix B (cf. Section 4.7.2).

Proposition 4.3. *For any level $\ell \in \mathbb{R}$ and intensity $\vartheta > 0$, the maturity-randomized first-passage probability contributed by the diffusion part, $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$, satisfies the following Cauchy problem:*

$$\mathcal{A}_X \mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta, x; \ell) = \vartheta\, \mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta, x; \ell), \qquad \text{on } x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}}, \tag{4.3.30}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta, x; \ell) = 1, \qquad \text{on } x = \ell, \tag{4.3.31}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta, x; \ell) = 0, \qquad \text{on } x \in \mathcal{H}_\ell^{\pm}. \tag{4.3.32}$$

Similarly, the maturity-randomized first-passage probability contributed by jumps, $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$, solves, for any $\ell \in \mathbb{R}$ and $\vartheta > 0$, the Cauchy problem

$$\mathcal{A}_X \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta, x; \ell) = \vartheta\, \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta, x; \ell), \qquad \text{on } x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^{\pm}}, \tag{4.3.33}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta, x; \ell) = 0, \qquad \text{on } x = \ell, \tag{4.3.34}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta, x; \ell) = 1, \qquad \text{on } x \in \mathcal{H}_\ell^{\pm}. \tag{4.3.35}$$

Compared with the results in Proposition 4.2, Proposition 4.3 offers substantially simpler characterizations. In particular, applying the Laplace-Carson operator to the first-passage probabilities reduces the complexity of the respective problems by transforming the PIDE characterizations of Proposition 4.2 into corresponding OIDE characterizations. Under certain Lévy dynamics $(X_t)_{t\geq 0}$ the resulting problems (4.3.30)-(4.3.32) and (4.3.33)-(4.3.35) even have a simple analytical solution. This is in particular true for the class of hyper-exponential jump-diffusions that is discussed in Section 4.3.4.

4.3.3 Intra-Horizon Risk and Risk Contributions

The analysis developed in the previous sections provided a decomposition of diffusion and jump contributions embodied in first-passage probabilities. Since both the intra-horizon value at risk and the intra-horizon expected shortfall can be fully characterized based on first-passage probabilities (cf. Section 4.2), these last results can be further extended to infer diffusion and jump risk contributions to the intra-horizon risk measures under consideration. This is discussed next.

We start by introducing risk contributions for the intra-horizon value at risk. Here, we follow the ideas in [LV20] and understand the diffusion and jump risk contributions as the proportions of the $iV@R$-first-passage probability contributed by the respective components, i.e. we define, for $\alpha \in (0,1)$ and a (discounted) profit and loss process $(\mathcal{P\&L}_t)_{t\in[0,T]}$ satisfying the dynamics specified in either *Scenario 1* or *Scenario 2*, the diffusion and jump risk contribution inherent to the level-α intra-horizon value at risk over the time interval $[0,T]$, $\mathcal{R}^{\mathcal{D}}_{iV@R}(\mathcal{P\&L};\alpha,T)$ and $\mathcal{R}^{\mathcal{J}}_{iV@R}(\mathcal{P\&L};\alpha,T)$ respectively, via

$$\mathcal{R}^{\mathcal{D}}_{iV@R}(\mathcal{P\&L};\alpha,T) := \begin{cases} \dfrac{u_X^{\mathcal{E}_0^-}\big(T, z; -iV@R_{\alpha,T}(\mathcal{P\&L})\big)}{u_X^-\big(T, z; -iV@R_{\alpha,T}(\mathcal{P\&L})\big)}, & \text{for } \textit{Scenario 1}, \\[2ex] \dfrac{u_X^{\mathcal{E}_0^-}\big(T, \log(z_2+z); \log\big(z_2 - iV@R_{\alpha,T}(\mathcal{P\&L})\big)\big)}{u_X^-\big(T, \log(z_2+z); \log\big(z_2 - iV@R_{\alpha,T}(\mathcal{P\&L})\big)\big)}, & \text{for a long position in } \textit{Scenario 2}, \\[2ex] \dfrac{u_X^{\mathcal{E}_0^+}\big(T, \log(z_2-z); \log\big(z_2 + iV@R_{\alpha,T}(\mathcal{P\&L})\big)\big)}{u_X^+\big(T, \log(z_2-z); \log\big(z_2 + iV@R_{\alpha,T}(\mathcal{P\&L})\big)\big)}, & \text{for a short position in } \textit{Scenario 2}, \end{cases} \tag{4.3.36}$$

and

$$\mathcal{R}^{\mathcal{J}}_{iV@R}(\mathcal{P}\&\mathcal{L};\alpha,T) := \begin{cases} \dfrac{u_X^{\mathcal{E}_{\mathcal{J}}^-}\big(T,z;-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)}{u_X^-\big(T,z;-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)}, & \text{for } \textit{Scenario 1}, \\ \dfrac{u_X^{\mathcal{E}_{\mathcal{J}}^-}\big(T,\log(z_2+z);\log\big(z_2-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\big)}{u_X^-\big(T,\log(z_2+z);\log\big(z_2-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\big)}, & \text{for a long position in } \textit{Scenario 2}, \\ \dfrac{u_X^{\mathcal{E}_{\mathcal{J}}^+}\big(T,\log(z_2-z);\log\big(z_2+iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\big)}{u_X^+\big(T,\log(z_2-z);\log\big(z_2+iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\big)}, & \text{for a short position in } \textit{Scenario 2}. \end{cases} \tag{4.3.37}$$

Defining risk contributions embodied in the intra-horizon expected shortfall can be done via similar techniques and is closely linked to the computation of risk contributions for the intra-horizon value at risk. Indeed, from Proposition 4.1 we already know that (given the intra-horizon value at risk to a certain level) the difference between intra-horizon expected shortfall and intra-horizon value at risk consists in an integral over first-passage probabilities that is given by

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) - iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L}) = \frac{1}{\alpha}\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u(T,z;\ell)\,d\ell, \tag{4.3.38}$$

where $u(T,z;\ell)$ is specified in each scenario according to Relation (4.3.6). Therefore, the diffusion and jump risk contributions inherent to the intra-horizon expected shortfall can be divided into two parts: the respective risk contributions in the intra-horizon value at risk and those of the remaining integral (4.3.38). For the latter – which can be interpreted as an average conditional excess intra-horizon tail loss – diffusion and jump risk contributions can be defined as the proportions of the integral contributed by the respective components, i.e. one recovers, for $\alpha \in (0,1)$ and a (discounted) profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$ satisfying the dynamics specified in either *Scenario 1* or *Scenario 2*, the diffusion and jump risk contribution inherent to the integral part (4.3.38), $\mathcal{R}^{\mathcal{D}}_{\mathcal{I}}(\mathcal{P}\&\mathcal{L};\alpha,T)$ and $\mathcal{R}^{\mathcal{J}}_{\mathcal{I}}(\mathcal{P}\&\mathcal{L};\alpha,T)$ respectively, via

$$\mathcal{R}^{\mathcal{D}}_{\mathcal{I}}(\mathcal{P}\&\mathcal{L};\alpha,T) = \begin{cases} \dfrac{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_0^-}(T,z;\ell)\,d\ell}{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^-(T,z;\ell)\,d\ell}, & \text{for } \textit{Scenario 1}, \\ \dfrac{\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_0^-}(T,\log(z_2+z);\log(z_2+\ell))\,d\ell}{\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^-(T,\log(z_2+z);\log(z_2+\ell))\,d\ell}, & \text{for a long position in } \textit{Scenario 2}, \\ \dfrac{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_0^+}(T,\log(z_2-z);\log(z_2-\ell))\,d\ell}{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^+(T,\log(z_2-z);\log(z_2-\ell))\,d\ell}, & \text{for a short position in } \textit{Scenario 2}, \end{cases} \tag{4.3.39}$$

and

$$\mathcal{R}_{\mathcal{I}}^{\mathcal{J}}(\mathcal{P}\&\mathcal{L};\alpha,T)=\begin{cases}\dfrac{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_{\mathcal{J}}^{-}}(T,z;\ell)\,d\ell}{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{-}(T,z;\ell)\,d\ell}, & \text{for } \textit{Scenario 1},\\[2ex] \dfrac{\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_{\mathcal{J}}^{-}}(T,\log(z_2+z);\log(z_2+\ell))\,d\ell}{\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{-}(T,\log(z_2+z);\log(z_2+\ell))\,d\ell}, & \text{for a long position in } \textit{Scenario 2},\\[2ex] \dfrac{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_{\mathcal{J}}^{+}}(T,\log(z_2-z);\log(z_2-\ell))\,d\ell}{\int_{-\infty}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{+}(T,\log(z_2-z);\log(z_2-\ell))\,d\ell}, & \text{for a short position in } \textit{Scenario 2}.\end{cases} \tag{4.3.40}$$

Finally, using these definitions, the diffusion and jump risk contributions inherent to the level-α intra-horizon expected shortfall over the time interval $[0,T]$, $\mathcal{R}_{iES}^{\mathcal{D}}(\mathcal{P}\&\mathcal{L};\alpha,T)$ and $\mathcal{R}_{iES}^{\mathcal{J}}(\mathcal{P}\&\mathcal{L};\alpha,T)$ respectively, can be recovered as weighted sums of the corresponding contributions for the intra-horizon value at risk and the integral part (4.3.38), i.e. as

$$\mathcal{R}_{iES}^{\mathcal{D}}(\mathcal{P}\&\mathcal{L};\alpha,T)=\big(1-\omega_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\cdot\mathcal{R}_{\mathcal{I}}^{\mathcal{D}}(\mathcal{P}\&\mathcal{L};\alpha,T)+\omega_{\alpha,T}(\mathcal{P}\&\mathcal{L})\cdot\mathcal{R}_{iV@R}^{\mathcal{D}}(\mathcal{P}\&\mathcal{L};\alpha,T), \tag{4.3.41}$$

$$\mathcal{R}_{iES}^{\mathcal{J}}(\mathcal{P}\&\mathcal{L};\alpha,T)=\big(1-\omega_{\alpha,T}(\mathcal{P}\&\mathcal{L})\big)\cdot\mathcal{R}_{\mathcal{I}}^{\mathcal{J}}(\mathcal{P}\&\mathcal{L};\alpha,T)+\omega_{\alpha,T}(\mathcal{P}\&\mathcal{L})\cdot\mathcal{R}_{iV@R}^{\mathcal{J}}(\mathcal{P}\&\mathcal{L};\alpha,T), \tag{4.3.42}$$

where $\omega_{\alpha,T}(\mathcal{P}\&\mathcal{L}):=\frac{iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})}{iES_{\alpha,T}(\mathcal{P}\&\mathcal{L})}$ denotes the contribution of the intra-horizon value at risk to the intra-horizon expected shortfall. We will come back to this decomposition when discussing numerical results in Section 4.5.

4.3.4 First-Passage Decomposition and Intra-Horizon Risk under Hyper-Exponential Jump-Diffusions

Having elaborated on our core intra-horizon risk measurement approach under the general Lévy framework, we next discuss (semi-)analytical expressions for hyper-exponential jump-diffusion processes. These results are particularly interesting since they subsequently allow for an approximate, though arbitrarily precise semi-analytical measurement of intra-horizon risk within the important class of Lévy processes having a completely monotone jump density. We will further develop this point in Section 4.4 and lastly provide an application of this approximate approach in the numerical analysis of Section 4.5.

4.3.4.1 Generalities on Hyper-Exponential Jump-Diffusions

We recall that a hyper-exponential jump-diffusion process $(X_t)_{t\geq 0}$ is a Lévy process that combines a Brownian diffusion with hyper-exponentially distributed jumps. This process has the usual jump-diffusion structure, i.e. it can be characterized on a filtered probability space $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P}^X)$ via

$$X_t=\mu t+\sigma_X W_t+\sum_{i=1}^{N_t}J_i,\qquad \text{for}\quad t\geq 0, \tag{4.3.43}$$

where $(W_t)_{t\geq 0}$ denotes an $\mathbf{F}$-Brownian motion and $(N_t)_{t\geq 0}$ is an $\mathbf{F}$-Poisson process that has intensity parameter $\lambda>0$. The constants $\mu\in\mathbb{R}$ and $\sigma_X\geq 0$ denote the drift and volatility parameters of the diffusion part respectively. Additionally, the jumps $(J_i)_{i\in\mathbb{N}}$ are assumed to be independent of $(N_t)_{t\geq 0}$ and to form a sequence of independent and identically distributed random variables following a hyper-exponential distribution, i.e. their (common) density function $f_{J_1}(\cdot)$ is given by

$$f_{J_1}(y) = \sum_{i=1}^{m} p_i \xi_i e^{-\xi_i y} \mathbb{1}_{\{y\geq 0\}} + \sum_{j=1}^{n} q_j \eta_j e^{\eta_j y} \mathbb{1}_{\{y<0\}}, \tag{4.3.44}$$

where $p_i > 0$ and $\xi_i > 1$ for $i \in \{1, \ldots, m\}$ and $q_j > 0$ and $\eta_j > 0$ for $j \in \{1, \ldots, n\}$. Here, the parameters $(p_i)_{i\in\{1,\ldots,m\}}$ and $(q_j)_{j\in\{1,\ldots,n\}}$ represent the proportion of jumps that are attributed to particular jump types and are therefore assumed to satisfy the condition $\sum\limits_{i=1}^{m} p_i + \sum\limits_{j=1}^{n} q_j = 1$. Finally, we will always assume that the intensity parameters $(\xi_i)_{i\in\{1,\ldots,m\}}$ and $(\eta_j)_{j\in\{1,\ldots,n\}}$ are ordered in the sense that

$$\xi_1 < \xi_2 < \cdots < \xi_m \qquad \text{and} \qquad \eta_1 < \eta_2 < \cdots < \eta_n \tag{4.3.45}$$

and note that this does not consist in a loss of generality.

As special class of Lévy processes, hyper-exponential jump-diffusions can be equivalently characterized in terms of their Lévy triplet $\big(b_X, \sigma_X^2, \Pi_X\big)$, where b_X and Π_X are then obtained as

$$b_X := \mu + \int_{\{|y|\leq 1\}} y \, \Pi_X(dy) \qquad \text{and} \qquad \Pi_X(dy) := \lambda f_{J_1}(y)dy. \tag{4.3.46}$$

Using these results, their Lévy exponent, $\Psi_X(\cdot)$, is easily obtained via (4.3.1), as

$$\Psi_X(\theta) = -i\mu\theta + \frac{1}{2}\sigma_X^2\theta^2 - \lambda \left(\sum_{i=1}^{m} \frac{p_i\xi_i}{\xi_i - i\theta} + \sum_{j=1}^{n} \frac{q_j\eta_j}{\eta_j + i\theta} - 1 \right). \tag{4.3.47}$$

Similarly, the corresponding Laplace exponent, $\Phi_X(\cdot)$, is well-defined for $\theta \in (-\eta_1, \xi_1)$ and equals

$$\Phi_X(\theta) = \mu\theta + \frac{1}{2}\sigma_X^2\theta^2 + \lambda \left(\sum_{i=1}^{m} \frac{p_i\xi_i}{\xi_i - \theta} + \sum_{j=1}^{n} \frac{q_j\eta_j}{\eta_j + \theta} - 1 \right). \tag{4.3.48}$$

In what follows, we will consider the Laplace exponent as standalone function on the extended real domain $\Phi_X : \mathbb{R} \setminus \{\xi_1, \ldots, \xi_m, -\eta_1, \ldots, -\eta_n\} \to \mathbb{R}$. This quantity will play a central role in the upcoming derivations. In fact, many distributional properties of hyper-exponential jump-diffusion processes (and of their generalizations) are closely linked to the roots of the equation $\Phi_X(\theta) = \alpha$, for $\alpha \geq 0$. This was already used in diverse articles dealing with option pricing and risk management within the class of mixed-exponential jump-diffusion processes (cf. among others [Ca09], [CCW09], [CK11], [CK12]). In this context, the following important lemma was partly derived in [Ca09] under hyper-exponential jump-diffusion models. Since the proof of all the remaining statements do not substantially differ from the results derived in [Ca09], the reader is referred to the arguments provided in this article.

Lemma 4.2. *For $\Phi_X(\cdot)$ defined as in (4.3.48) and any $\alpha > 0$, the following holds:*

1. *If $\sigma_X \neq 0$, the equation $\Phi_X(\theta) = \alpha$ has $(m+n+2)$ real roots $\beta_{1,\alpha}, \ldots, \beta_{m+1,\alpha}$ and $\gamma_{1,\alpha}, \ldots, \gamma_{n+1,\alpha}$ that satisfy*

$$-\infty < \gamma_{n+1,\alpha} < -\eta_n < \gamma_{n,\alpha} < -\eta_{n-1} < \cdots < \gamma_{2,\alpha} < -\eta_1 < \gamma_{1,\alpha} < 0, \tag{4.3.49}$$

$$0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \cdots < \xi_{m-1} < \beta_{m,\alpha} < \xi_m < \beta_{m+1,\alpha} < \infty. \tag{4.3.50}$$

2. *If $\sigma_X = 0$ and $\mu \neq 0$ the equation $\Phi_X(\theta) = \alpha$ has $(m+n+1)$ real roots. Specifically,*

   - *if $\mu > 0$, there are $m+1$ positive roots $\beta_{1,\alpha}, \dots, \beta_{m+1,\alpha}$ and n negative roots $\gamma_{1,\alpha}, \dots, \gamma_{n,\alpha}$ that satisfy*

$$-\infty < -\eta_n < \gamma_{n,\alpha} < -\eta_{n-1} < \cdots < \gamma_{2,\alpha} < -\eta_1 < \gamma_{1,\alpha} < 0, \tag{4.3.51}$$
$$0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \cdots < \xi_{m-1} < \beta_{m,\alpha} < \xi_m < \beta_{m+1,\alpha} < \infty. \tag{4.3.52}$$

   - *if $\mu < 0$, there are m positive roots $\beta_{1,\alpha}, \dots, \beta_{m,\alpha}$ and $n+1$ negative roots $\gamma_{1,\alpha}, \dots, \gamma_{n+1,\alpha}$ that satisfy*

$$-\infty < \gamma_{n+1,\alpha} < -\eta_n < \gamma_{n,\alpha} < -\eta_{n-1} < \cdots < \gamma_{2,\alpha} < -\eta_1 < \gamma_{1,\alpha} < 0, \tag{4.3.53}$$
$$0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \cdots < \xi_{m-1} < \beta_{m,\alpha} < \xi_m < \infty. \tag{4.3.54}$$

3. *If $\sigma_X = 0$ and $\mu = 0$ the equation $\Phi_X(\theta) = \alpha$ has $(m+n)$ real roots $\beta_{1,\alpha}, \dots, \beta_{m,\alpha}$ and $\gamma_{1,\alpha}, \dots, \gamma_{n,\alpha}$ that satisfy*

$$-\infty < -\eta_n < \gamma_{n,\alpha} < -\eta_{n-1} < \cdots < \gamma_{2,\alpha} < -\eta_1 < \gamma_{1,\alpha} < 0, \tag{4.3.55}$$
$$0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \cdots < \xi_{m-1} < \beta_{m,\alpha} < \xi_m < \infty. \tag{4.3.56}$$

At this point, we should mention that the roots in Lemma 4.2 are only known in analytical form in very few cases. Nevertheless, this does not impact the importance and practicability of Lemma 4.2 since the roots can be anyway recovered using standard numerical techniques.

4.3.4.2 Maturity-Randomization and OIDEs

We turn back to the OIDE characterizations of Proposition 4.3 and consider the respective problems (4.3.30)-(4.3.32) and (4.3.33)-(4.3.35) under hyper-exponential jump-diffusion processes with non-zero volatility parameter $\sigma_X \neq 0$. Switching to the case where $\sigma_X = 0$ does not fundamentally change the approach and only few, slight adaptions are needed. We will address some of these adaptions in Section 4.4, when discussing hyper-exponential jump-diffusion approximations to infinite-activity pure jump processes.

To start, we note that the infinitesimal generator (4.3.4) simplifies in this case to

$$\mathcal{A}_X V(\mathcal{T}, x) = \frac{1}{2}\sigma_X^2 \partial_x^2 V(\mathcal{T}, x) + \mu \partial_x V(\mathcal{T}, x) + \lambda \int_{\mathbb{R}} \big(V(\mathcal{T}, x+y) - V(\mathcal{T}, x) \big) f_{J_1}(y) dy, \tag{4.3.57}$$

which allows us, together with the properties of the hyper-exponential density $f_{J_1}(\cdot)$, to uniquely solve Problems (4.3.30)-(4.3.32) and (4.3.33)-(4.3.35) and to derive closed-form expressions for the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$. Specifically, we define for any $\vartheta > 0$ the $(m+1) \times (m+1)$ and $(n+1) \times (n+1)$ matrices $\underline{\mathbf{A}}_\vartheta$ and $\overline{\mathbf{A}}_\vartheta$ respectively via

$$\underline{\mathbf{A}}_\vartheta := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\xi_1}{\xi_1 - \beta_{1,\vartheta}} & \frac{\xi_1}{\xi_1 - \beta_{2,\vartheta}} & \cdots & \frac{\xi_1}{\xi_1 - \beta_{m+1,\vartheta}} \\ \frac{\xi_2}{\xi_2 - \beta_{1,\vartheta}} & \frac{\xi_2}{\xi_2 - \beta_{2,\vartheta}} & \cdots & \frac{\xi_2}{\xi_2 - \beta_{m+1,\vartheta}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\xi_m}{\xi_m - \beta_{1,\vartheta}} & \frac{\xi_m}{\xi_m - \beta_{2,\vartheta}} & \cdots & \frac{\xi_m}{\xi_m - \beta_{m+1,\vartheta}} \end{pmatrix}, \qquad \overline{\mathbf{A}}_\vartheta := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\eta_1}{\eta_1 + \gamma_{1,\vartheta}} & \frac{\eta_1}{\eta_1 + \gamma_{2,\vartheta}} & \cdots & \frac{\eta_1}{\eta_1 + \gamma_{n+1,\vartheta}} \\ \frac{\eta_2}{\eta_2 + \gamma_{1,\vartheta}} & \frac{\eta_2}{\eta_2 + \gamma_{2,\vartheta}} & \cdots & \frac{\eta_2}{\eta_2 + \gamma_{n+1,\vartheta}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_n}{\eta_n + \gamma_{1,\vartheta}} & \frac{\eta_n}{\eta_n + \gamma_{2,\vartheta}} & \cdots & \frac{\eta_n}{\eta_n + \gamma_{n+1,\vartheta}} \end{pmatrix}, \tag{4.3.58}$$

and observe that these matrices are invertible.[5] Additionally, we denote for any $k \in \mathbb{N}$ the $1 \times k$ vectors of zeros and ones as

$$\mathbf{0}_k := (\underbrace{0, \dots, 0}_{k}) \qquad \text{and} \qquad \mathbf{1}_k := (\underbrace{1, \dots, 1}_{k}). \tag{4.3.59}$$

Using the above notation, the following proposition can be derived. The proof is presented in Appendix C (cf. Section 4.7.3).

Proposition 4.4. *Assume that $(X_t)_{t\geq 0}$ follows a hyper-exponential jump-diffusion process with non-zero diffusion component, as described in (4.3.43), (4.3.44) with $\sigma_X \neq 0$. Then, for any level $\ell \in \mathbb{R}$ and intensity $\vartheta > 0$, the maturity-randomized upside first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^+}\big)(\cdot)$ take the form*

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta, x; \ell) = \begin{cases} 0, & x > \ell, \\ 1, & x = \ell, \\ \sum_{k=1}^{m+1} \underline{v}_{0,k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)}, & x < \ell, \end{cases} \quad \textit{and} \quad \mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^+}\big)(\vartheta, x; \ell) = \begin{cases} 1, & x > \ell, \\ 0, & x = \ell, \\ \sum_{k=1}^{m+1} \underline{v}_{\mathcal{J},k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)}, & x < \ell. \end{cases} \tag{4.3.60}$$

Here, $\underline{\mathbf{v}}_0 := \big(\underline{v}_{0,1}, \dots, \underline{v}_{0,m+1}\big)^\intercal$ and $\underline{\mathbf{v}}_\mathcal{J} := \big(\underline{v}_{\mathcal{J},1}, \dots, \underline{v}_{\mathcal{J},m+1}\big)^\intercal$ are weight vectors uniquely determined by the system of linear equations

$$\underline{\mathbf{A}}_\vartheta \underline{\mathbf{v}}_0 = (1, \mathbf{0}_m)^\intercal \qquad \textit{and} \qquad \underline{\mathbf{A}}_\vartheta \underline{\mathbf{v}}_\mathcal{J} = (0, \mathbf{1}_m)^\intercal. \tag{4.3.61}$$

Similarly, for $\ell \in \mathbb{R}$ and $\vartheta > 0$, the maturity-randomized downside first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^-}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^-}\big)(\cdot)$ are given by

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^-}\big)(\vartheta, x; \ell) = \begin{cases} \sum_{k=1}^{n+1} \overline{v}_{0,k}\, e^{\gamma_{k,\vartheta}\cdot(x-\ell)}, & x > \ell, \\ 1, & x = \ell, \\ 0, & x < \ell, \end{cases} \quad \textit{and} \quad \mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^-}\big)(\vartheta, x; \ell) = \begin{cases} \sum_{k=1}^{n+1} \overline{v}_{\mathcal{J},k}\, e^{\gamma_{k,\vartheta}\cdot(x-\ell)}, & x > \ell, \\ 0, & x = \ell, \\ 1, & x < \ell, \end{cases} \tag{4.3.62}$$

where, $\overline{\mathbf{v}}_0 := (\overline{v}_{0,1}, \dots, \overline{v}_{0,n+1})^\intercal$ and $\overline{\mathbf{v}}_\mathcal{J} := (\overline{v}_{\mathcal{J},1}, \dots, \overline{v}_{\mathcal{J},n+1})^\intercal$ are uniquely determined by the system of linear equations

$$\overline{\mathbf{A}}_\vartheta \overline{\mathbf{v}}_0 = (1, \mathbf{0}_n)^\intercal \qquad \textit{and} \qquad \overline{\mathbf{A}}_\vartheta \overline{\mathbf{v}}_\mathcal{J} = (0, \mathbf{1}_n)^\intercal. \tag{4.3.63}$$

Proposition 4.4 already provides an important analytical disentanglement of the diffusion and jump contributions underlying first-passage probabilities. Nevertheless, it may be additionally insightful to understand how the jump risk is further distributed across jump types. Under hyper-exponential jump-diffusion processes such a decomposition can be derived and this is discussed next. To start, we define for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ the following jump-events

$$\mathcal{E}_i^+ := \Big\{ X_{\tau_\ell^{X,+}} > \ell,\, J_{N_{\tau_\ell^{X,+}}} \sim \mathrm{Exp}(\xi_i) \Big\} \quad \text{and} \quad \mathcal{E}_j^- := \Big\{ X_{\tau_\ell^{X,-}} < \ell,\, J_{N_{\tau_\ell^{X,-}}} \sim \mathrm{Exp}(\eta_j) \Big\}, \tag{4.3.64}$$

and see that

$$\mathcal{E}_\mathcal{J}^+ \setminus \Big(\{X_0 > \ell\} \cup \{\tau_\ell^{X,+} = \infty\} \Big) = \bigcup_{i=1}^{m} \mathcal{E}_i^+ \quad \text{and} \quad \mathcal{E}_\mathcal{J}^- \setminus \Big(\{X_0 < \ell\} \cup \{\tau_\ell^{X,-} = \infty\} \Big) = \bigcup_{j=1}^{n} \mathcal{E}_j^-, \tag{4.3.65}$$

[5] The invertibility of $\underline{\mathbf{A}}_\vartheta$ and $\overline{\mathbf{A}}_\vartheta$ can be proved as in [CK11] and the reader is referred to this article.

i.e. we essentially decompose the first-passage events contributed by jumps in events that are triggered by jumps of certain types. Additionally, we note that the first-passage events $\mathcal{E}_i^+$ and $\mathcal{E}_j^-$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ are disjoint among each others. In particular, this gives that the first-passage probabilities contributed by jumps, $u_X^{\mathcal{E}_\mathcal{J}^\pm}(\cdot)$, and the respective maturity-randomized quantities, $\mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^\pm}\big)(\cdot)$, have, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \mathcal{H}_\ell^\pm$, and $\mathcal{T} \in [0, T]$ and $\vartheta > 0$ respectively the following decomposition

$$u_X^{\mathcal{E}_\mathcal{J}^+}(\mathcal{T}, x; \ell) = \sum_{i=1}^{m} u_X^{\mathcal{E}_i^+}(\mathcal{T}, x; \ell), \qquad u_X^{\mathcal{E}_\mathcal{J}^-}(\mathcal{T}, x; \ell) = \sum_{j=1}^{n} u_X^{\mathcal{E}_j^-}(\mathcal{T}, x; \ell), \tag{4.3.66}$$

$$\mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^+}\big)(\vartheta, x; \ell) = \sum_{i=1}^{m} \mathcal{LC}\big(u_X^{\mathcal{E}_i^+}\big)(\vartheta, x; \ell), \qquad \mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^-}\big)(\vartheta, x; \ell) = \sum_{j=1}^{n} \mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)(\vartheta, x; \ell), \tag{4.3.67}$$

where $u_X^{\mathcal{E}_i^+}(\mathcal{T}, x; \ell)$ for $i \in \{1, \ldots, m\}$ and $u_X^{\mathcal{E}_j^-}(\mathcal{T}, x; \ell)$ for $j \in \{1, \ldots, n\}$ refer to the functions defined by

$$u_X^{\mathcal{E}_i^+}(\mathcal{T}, x; \ell) := \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,+} \leq \mathcal{T}\} \cap \mathcal{E}_i^+}\right] \quad \text{and} \quad u_X^{\mathcal{E}_j^-}(\mathcal{T}, x; \ell) := \mathbb{E}_x^{\mathbb{P}^X}\left[\mathbb{1}_{\{\tau_\ell^{X,-} \leq \mathcal{T}\} \cap \mathcal{E}_j^-}\right]. \tag{4.3.68}$$

Next, we note as in [KW03] that the monotonicity of the cumulative distribution functions $t \mapsto u_X^{\mathcal{E}}(t, x; \ell)$ for $\mathcal{E} \in \{\mathcal{E}_0^\pm, \mathcal{E}_\mathcal{J}^\pm, \mathcal{E}_1^+, \ldots, \mathcal{E}_m^+, \mathcal{E}_1^-, \ldots, \mathcal{E}_n^-\}$ implies that we can rewrite each of the maturity-randomized versions $\mathcal{LC}(u_X^{\mathcal{E}})(\cdot)$, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^\pm}$ and $\vartheta > 0$, in the form

$$\begin{aligned}
\mathcal{LC}(u_X^{\mathcal{E}})(\vartheta, x; \ell) &:= \int_0^\infty \vartheta e^{-\vartheta t}\, u_X^{\mathcal{E}}(t, x; \ell) dt \\
&= \int_0^\infty e^{-\vartheta t}\, \partial_t u_X^{\mathcal{E}}(t, x; \ell) dt \\
&= \int_0^\infty e^{-\vartheta t}\, u_X^{\mathcal{E}}(dt, x; \ell) = \mathbb{E}_x^{\mathbb{P}^X}\left[e^{-\vartheta\, \tau_\ell^{X,\pm}}\, \mathbb{1}_{\mathcal{E}}\right],
\end{aligned} \tag{4.3.69}$$

where $\tau_\ell^{X,\pm} = \tau_\ell^{X,+}$ for $\mathcal{E} \in \{\mathcal{E}_1^+, \ldots, \mathcal{E}_m^+\}$ and $\tau_\ell^{X,\pm} = \tau_\ell^{X,-}$ for $\mathcal{E} \in \{\mathcal{E}_1^-, \ldots, \mathcal{E}_n^-\}$. Combining this representation with the fact that the overshoot distribution is conditionally memoryless and independent of the first-passage time, given that the overshoot is greater than zero and that the exponential type of the jump distribution is specified (cf. [Ca09]), finally allows us to arrive at the next proposition. The proof is provided in Appendix C (cf. Section 4.7.3).

Proposition 4.5. *Assume that $(X_t)_{t\geq 0}$ follows a hyper-exponential jump-diffusion process with non-zero diffusion component, as described in (4.3.43), (4.3.44) with $\sigma_X \neq 0$ and define for any level $\ell \in \mathbb{R}$, $x \in \mathbb{R}$ and intensity $\vartheta > 0$ the following vectors*

$$\underline{\mathbf{LC}}_{\vartheta,\ell}(x) := \Big(\mathcal{LC}(u_X^{\mathcal{E}_0^+})(\vartheta, x; \ell), \ldots, \mathcal{LC}(u_X^{\mathcal{E}_m^+})(\vartheta, x; \ell)\Big)^\intercal, \quad \underline{\mathbf{e}}_{\vartheta,\ell}(x) := \big(e^{\beta_{1,\vartheta}\cdot(x-\ell)}, \ldots, e^{\beta_{m+1,\vartheta}\cdot(x-\ell)}\big)^\intercal, \tag{4.3.70}$$

$$\overline{\mathbf{LC}}_{\vartheta,\ell}(x) := \Big(\mathcal{LC}(u_X^{\mathcal{E}_0^-})(\vartheta, x; \ell), \ldots, \mathcal{LC}(u_X^{\mathcal{E}_n^-})(\vartheta, x; \ell)\Big)^\intercal, \quad \overline{\mathbf{e}}_{\vartheta,\ell}(x) := \big(e^{\gamma_{1,\vartheta}\cdot(x-\ell)}, \ldots, e^{\gamma_{n+1,\vartheta}\cdot(x-\ell)}\big)^\intercal. \tag{4.3.71}$$

Then, for $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$, the vector of maturity-randomized upside jump contributions, $\underline{\mathbf{LC}}_{\vartheta,\ell}(x)$, is uniquely determined by the system of linear equations

$$\underline{\mathbf{A}}_\vartheta^\intercal \, \underline{\mathbf{LC}}_{\vartheta,\ell}(x) = \underline{\mathbf{e}}_{\vartheta,\ell}(x). \tag{4.3.72}$$

Similarly, for $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^-}$ the vector of maturity-randomized downside jump contributions, $\overline{\mathbf{LC}}_{\vartheta,\ell}(x)$, is uniquely determined by the system of linear equations

$$\overline{\mathbf{A}}_\vartheta^\intercal \, \overline{\mathbf{LC}}_{\vartheta,\ell}(x) = \overline{\mathbf{e}}_{\vartheta,\ell}(x). \tag{4.3.73}$$

Proposition 4.5 provides an implicit characterization of diffusion and jump contributions underlying (maturity-randomized) first-passage probabilities and already allows for a derivation of the full vectors $\underline{\mathbf{LC}}_{\vartheta,\ell}(x)$ or $\overline{\mathbf{LC}}_{\vartheta,\ell}(x)$ using standard numerical methods. Nevertheless, the systems (4.3.72) and (4.3.73) can be explicitly solved to derive analytical expressions for each of the functions $\mathcal{LC}(u_X^{\mathcal{E}})(\cdot)$ with $\mathcal{E} \in \{\mathcal{E}_0^+, \mathcal{E}_1^+, \ldots, \mathcal{E}_m^+\}$ or $\mathcal{E} \in \{\mathcal{E}_0^-, \mathcal{E}_1^-, \ldots, \mathcal{E}_n^-\}$. This was already derived in a different context in [CYY13]. In particular, their results can be refined to arrive at the following useful proposition.

Proposition 4.6. *Assume that $(X_t)_{t\geq 0}$ follows a hyper-exponential jump-diffusion process with non-zero diffusion component, as described in (4.3.43), (4.3.44) with $\sigma_X \neq 0$. Then, for any level $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$ and intensity $\vartheta > 0$ we have that*

$$\mathcal{LC}\big(u_X^{\mathcal{E}_i^+}\big)(\vartheta, x; \ell) = \sum_{k=1}^{m+1} v_{\mathcal{E}_i^+,k} \cdot e^{\beta_{k,\vartheta}\cdot(x-\ell)}, \qquad i \in \{1, \ldots, m\}, \tag{4.3.74}$$

where, for $k \in \{1, \ldots, m+1\}$, the coefficients $v_{\mathcal{E}_i^+,k}$ can be expressed in terms of the coefficients $\underline{v}_{0,k}$ by

$$v_{\mathcal{E}_i^+,1} = -\frac{1}{\xi_i}\frac{\mathbf{C}_\vartheta^+(\xi_i)}{\big(\mathbf{B}^+\big)'(\xi_i)}\,\underline{v}_{0,1}, \qquad \text{and} \qquad v_{\mathcal{E}_i^+,k} = -\frac{1}{\xi_i}\frac{\mathbf{C}_\vartheta^+(\xi_i)}{\big(\mathbf{B}^+\big)'(\xi_i)}\,\underline{v}_{0,k} + \frac{1}{\xi_i} d_{k,i}^+, \quad k \in \{2, \ldots, m+1\}, \tag{4.3.75}$$

with

$$\mathbf{B}^+(x) := \prod_{s=1}^{m}\big(\xi_s - x\big), \qquad \mathbf{C}_\vartheta^+(x) := \prod_{s=2}^{m+1}\big(\beta_{s,\vartheta} - x\big), \tag{4.3.76}$$

and

$$d_{i,j}^+ := -\frac{\mathbf{B}^+(\beta_{i,\vartheta})\,\mathbf{C}_\vartheta^+(\xi_j)}{(\xi_j - \beta_{i,\vartheta})\,\big(\mathbf{B}^+\big)'(\xi_j)\,\big(\mathbf{C}_\vartheta^+\big)'(\beta_{i,\vartheta})}, \qquad i \in \{2, \ldots, m+1\}, \;\; j \in \{1, \ldots, m\}. \tag{4.3.77}$$

Similarly, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^-}$ and $\vartheta > 0$ we have that

$$\mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)(\vartheta, x; \ell) = \sum_{k=1}^{n+1} v_{\mathcal{E}_j^-,k} \cdot e^{\gamma_{k,\vartheta}\cdot(x-\ell)}, \qquad j \in \{1, \ldots, n\}, \tag{4.3.78}$$

where the coefficients $v_{\mathcal{E}_j^-,k}$ are given, for $k \in \{1, \ldots, n+1\}$, by

$$v_{\mathcal{E}_j^-,1} = (-1)^n\,\frac{1}{\eta_j}\frac{\mathbf{C}_\vartheta^-(\eta_j)}{\big(\mathbf{B}^-\big)'(-\eta_j)}\,\overline{v}_{0,1}, \qquad \text{and} \qquad v_{\mathcal{E}_j^-,k} = (-1)^n\,\frac{1}{\eta_j}\frac{\mathbf{C}_\vartheta^-(\eta_j)}{\big(\mathbf{B}^-\big)'(-\eta_j)}\,\overline{v}_{0,k} + \frac{1}{\eta_j} d_{k,j}^-, \quad k \in \{2, \ldots, n+1\}, \tag{4.3.79}$$

with

$$\mathbf{B}^{-}(x) := \prod_{s=1}^{n} \big(\eta_s + x\big), \qquad \mathbf{C}_{\vartheta}^{-}(x) := \prod_{s=2}^{n+1} \big(\gamma_{s,\vartheta} + x\big), \tag{4.3.80}$$

and

$$d_{i,j}^{-} := \frac{\mathbf{B}^{-}(\gamma_{i,\vartheta})\, \mathbf{C}_{\vartheta}^{-}(\eta_j)}{(\eta_j + \gamma_{i,\vartheta})\, \big(\mathbf{B}^{-}\big)'(-\eta_j)\, \big(\mathbf{C}_{\vartheta}^{-}\big)'(-\gamma_{i,\vartheta})}, \qquad i \in \{2, \ldots, n+1\},\;\; j \in \{1, \ldots, n\}. \tag{4.3.81}$$

<u>**Remark 4.4.**</u>

i) We re-emphasize that the (full) results in Proposition 4.4, Proposition 4.5 and Proposition 4.6 only hold under non-zero diffusion component. In fact, when $\sigma_X = 0$ the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\pm}\big)(\cdot)$ reduce (for $x \neq \ell$) to the jump contributions $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$ and the finite activity of the underlying jump process implies that the continuous-fit conditions

$$\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{+}}\big)(\vartheta, \ell-; \ell) = 0 \qquad \text{and} \qquad \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{-}}\big)(\vartheta, \ell+; \ell) = 0$$

do not anymore hold. In addition, in view of Lemma 4.2, the matrices defined in (4.3.58) may have to be replaced by corresponding $(m+1) \times m$, $(n+1) \times n$, $m \times m$ or $n \times n$ matrices. Therefore, the resulting systems of equations (4.3.61), (4.3.63) and (4.3.72), (4.3.73) need to be adjusted accordingly and this may finally impact our derivations in Proposition 4.6. We will deal with these adaptions in more details in Section 4.4.

ii) In addition to obtaining (semi-)analytical expressions for $\mathcal{LC}(u_X^{\mathcal{E}})(\cdot)$ with $\mathcal{E} \in \{\mathcal{E}_0^{+}, \mathcal{E}_1^{+}, \ldots, \mathcal{E}_m^{+}\}$ or $\mathcal{E} \in \{\mathcal{E}_0^{-}, \mathcal{E}_1^{-}, \ldots, \mathcal{E}_n^{-}\}$, Proposition 4.6 reveals, together with (4.3.60) and (4.3.62), that for any values $x \in \mathbb{R} \setminus \overline{\mathcal{H}_{\ell}^{\pm}}$ and intensity $\vartheta > 0$ the functions $\ell \mapsto \mathcal{LC}(u_X^{\mathcal{E}})(\vartheta, x; \ell)$ with $\mathcal{E} \in \{\mathcal{E}_0^{+}, \mathcal{E}_1^{+}, \ldots, \mathcal{E}_m^{+}\}$ or $\mathcal{E} \in \{\mathcal{E}_0^{-}, \mathcal{E}_1^{-}, \ldots, \mathcal{E}_n^{-}\}$ consist in linear combinations of exponentials. We will combine this particularly simple form with the structure of the Gaver-Stehfest algorithm in Section 4.5 to derive a simple inversion algorithm for the integral part of Proposition 4.1.

♦

4.4 Approximating Models with Jumps via Hyper-Exponential Jump-Diffusions

In this section, we complement the theory developed in the previous parts by discussing hyper-exponential approximations to (infinite-activity) pure jump processes having a completely monotone jump density. We slightly adapt the approach followed in [AMP07], [JP10], briefly discuss the resulting approximations and comment on how they relate to our final aim of intra-horizon risk quantification. The results will play a central role in the upcoming numerical analysis of Section 4.5.

4.4.1 General Approximation Scheme

To start, we recall that a one-sided density $f : [0, \infty) \to \mathbb{R}$ is said to be completely monotone if for any $k \in \mathbb{N}$ its k-th derivative $f^{(k)}(\cdot)$ exists and $(-1)^k f^{(k)}(x) \geq 0$ holds on $x \in (0, \infty)$. Additionally, we recall that when dealing instead with a two-sided density $g : \mathbb{R} \to \mathbb{R}$ this definition naturally extends by requiring that both of the functions $x \mapsto g(x)\,\mathbb{1}_{[0,\infty)}(x)$ and $x \mapsto g(-x)\,\mathbb{1}_{[0,\infty)}(x)$ are completely monotone. It can be shown that many Lévy processes employed in financial modeling have a completely monotone jump density $\pi_X(\cdot)$, i.e. that their intensity measure Π_X takes the particular form

$$\Pi_X(dy) = \pi_X(y)\,dy \tag{4.4.1}$$

with $\pi_X(\cdot)$ being a (two-sided) completely monotone density. This includes among others hyper-exponential jump-diffusion models (cf. [Ko02], [Ca09]), Normal Inverse Gaussian (NIG) processes (cf. [BN97]) as well as the whole class of stable and tempered stable processes (cf. [KT13], [HK16]), containing the very popular Variance-Gamma (VG) (cf. [MS90], [MCC98]) and Carr-Geman-Madan-Yor (CGMY) models (cf. [CGMY02]). We are going to deal with some of these dynamics in Section 4.5.

In view of Bernstein's theorem, a Lévy process $(X_t)_{t\geq 0}$ has a completely monotone jump density if and only if its density can be decomposed as

$$\pi_X(y) = \mathbb{1}_{(0,\infty)}(y)\int_0^\infty e^{-uy}\,\mu^+(du) + \mathbb{1}_{(-\infty,0)}(y)\int_{-\infty}^0 e^{-|vy|}\,\mu^-(dv), \tag{4.4.2}$$

where $\mu^+(\cdot)$ and $\mu^-(\cdot)$ are (non-negative and finite) measures defined on $(0,\infty)$ and $(-\infty,0)$, respectively. In particular, discretizing these integrals allows to approximate the completely monotone density $\pi_X(\cdot)$ by a sequence $\big(\pi_X^{(n)}(\cdot)\big)_{n\in\mathbb{N}}$ of densities having the form

$$\pi_X^{(n)}(y) := \Lambda_n f_n(y), \qquad n \in \mathbb{N}, \tag{4.4.3}$$

where $f_n(\cdot), n \in \mathbb{N}$, are hyper-exponential densities defined, for partitions $(u_i^{(n)})_{i\in\{0,\dots,N_n\}}$ and $(v_j^{(n)})_{j\in\{0,\dots,M_n\}}$ of the sets $(0,\infty)$ and $(-\infty,0)$, respectively[6], having vanishing mesh[7], by

$$f_n(y) := \sum_{i=1}^{N_n} p_i^{(n)}\tilde{u}_i^{(n)} e^{-\tilde{u}_i^{(n)}y}\,\mathbb{1}_{\{y\geq 0\}} + \sum_{j=1}^{M_n} q_j^{(n)}|\tilde{v}_j^{(n)}|\,e^{|\tilde{v}_j^{(n)}|\,y}\,\mathbb{1}_{\{y<0\}}, \tag{4.4.4}$$

and, for $n \in \mathbb{N}$,

$$\tilde{u}_i^{(n)} := \frac{1}{2}\left(u_{i-1}^{(n)} + u_i^{(n)}\right),\ \ i = 1,\dots,N_n, \qquad \tilde{v}_j^{(n)} := \frac{1}{2}\left(v_{j-1}^{(n)} + v_j^{(n)}\right),\ \ j = 1,\dots,M_n, \tag{4.4.5}$$

$$p_i^{(n)} := \frac{\mu^+\big(\big[u_{i-1}^{(n)}, u_i^{(n)}\big)\big)}{\Lambda_n\cdot\tilde{u}_i^{(n)}},\ \ i = 1,\dots,N_n, \qquad q_j^{(n)} := \frac{\mu^-\big(\big(v_{j-1}^{(n)}, v_j^{(n)}\big]\big)}{\Lambda_n\cdot|\tilde{v}_j^{(n)}|},\ \ j = 1,\dots,M_n, \tag{4.4.6}$$

$$\Lambda_n := \sum_{i=1}^{N_n}\frac{\mu^+\big(\big[u_{i-1}^{(n)}, u_i^{(n)}\big)\big)}{\tilde{u}_i^{(n)}} + \sum_{j=1}^{M_n}\frac{\mu^-\big(\big(v_{j-1}^{(n)}, v_j^{(n)}\big]\big)}{|\tilde{v}_j^{(n)}|}. \tag{4.4.7}$$

This intuitive idea can be further developed and hyper-exponential jump-diffusion approximations to Lévy processes having completely monotone jumps can be derived. For this purpose, let $(X_t)_{t\geq 0}$ be such a process whose Lévy triplet is denoted by $\big(b_X, \sigma_X^2, \Pi_X\big)$. Then, for any sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers

[6] Our convention is that the elements of the partitions are increasing in their index, i.e. we assume that for any $n \in \mathbb{N}$ the relations $0 < |v_{M_n}^{(n)}| < \ldots < |v_0^{(n)}| < \infty$ and $0 < u_0^{(n)} < \ldots < u_{N_n}^{(n)} < \infty$ hold.

[7] Recall that the mesh of a partition $(u_i^{(n)})_{i\in\{0,\dots,N_n\}}$ is defined by $\max\limits_{1\leq i\leq N_n} |u_i^{(n)} - u_{i-1}^{(n)}|$.

converging to zero corresponding sequences of partitions $\big((u_i^{(n)})_{i\in\{0,\dots,N_n\}}\big)_{n\in\mathbb{N}}$ and $\big((v_j^{(n)})_{j\in\{0,\dots,M_n\}}\big)_{n\in\mathbb{N}}$ can be constructed such that the following conditions hold for each $n \in \mathbb{N}$:

$$\int\limits_{(-\infty,v_0^{(n)}]\cup[u_{N_n}^{(n)},\infty)} \pi_X(y)\,dy < \epsilon_n, \qquad \int\limits_{(v_0^{(n)},u_{N_n}^{(n)})\setminus(v_{M_n}^{(n)},u_0^{(n)})} \big(\pi_X(y) - \pi_X^{(n)}(y)\big)^2\,dy < \epsilon_n, \tag{4.4.8}$$

$$\int\limits_{v_{M_n}^{(n)}}^{u_0^{(n)}} y^2\big(\pi_X(y) - \pi_X^{(n)}(y)\big)\,dy < \epsilon_n. \tag{4.4.9}$$

In particular, Requirement (4.4.9) can be fulfilled since any Lévy process' intensity measure satisfies

$$\int\limits_{\mathbb{R}} \big(1 \wedge y^2\big)\,\Pi_X(dy) < \infty. \tag{4.4.10}$$

We denote by $(X_t^n)_{t\geq 0}$ the resulting, approximating Lévy process having jump density $\pi_X^{(n)}(\cdot)$, Gaussian parameter $\sigma_n^2 := \sigma_X^2$ and drift $b_n \in \mathbb{R}$ defined by the identity $\Phi_{X^n}(1) = \Phi_X(1)$. Then, following the lines of argument in [JP10] allows to see that the sequence of approximating processes constructed this way, $\big((X_t^n)_{t\geq 0}\big)_{n\in\mathbb{N}}$, converges weakly, in the Skorokhod topology, to the true process $(X_t)_{t\geq 0}$ and additionally that for any $\mathcal{T} \geq 0$, $x \in \mathbb{R}$ and $\ell \in \mathbb{R}$

$$\mathbb{P}_x^{X^n}\left(\tau_\ell^{X^n,\pm} \leq \mathcal{T}\right) \to \mathbb{P}_x^{X}\left(\tau_\ell^{X,\pm} \leq \mathcal{T}\right), \qquad n \to \infty. \tag{4.4.11}$$

In view of our general intra-horizon risk measurement approach, the latter convergence has an important implication. Not only can Lévy processes with completely monotone jumps be approximated by hyper-exponential jump-diffusion models, but the same approximating sequence can be also used for intra-horizon risk quantification. We will rely on this approach in Section 4.5, when discussing the intra-horizon risk inherent to certain infinite-activity pure jump Lévy dynamics, i.e. we will derive intra-horizon risk results to these Lévy models by relying on their hyper-exponential approximations $(X_t^n)_{t\geq 0}$ with $\sigma_n^2 = \sigma_X^2 = 0$. However, since our main results in Section 4.3.4 made explicitly use of the assumption $\sigma_X \neq 0$, a few adaptions need to be discussed. This is the content of the next section.

Remark 4.5.

Although our general approximation scheme relies on ideas similarly employed in [AMP07] and [JP10], the resulting approximating processes $\big((X_t^n)_{t\geq 0}\big)_{n\in\mathbb{N}}$ substantially deviate in their structure from the ones presented in these papers. This comes from the fact that the authors in [AMP07] and [JP10] choose to aggregate small jumps into an additional diffusion factor by taking

$$\sigma_n^2 := \sigma_X^2 + \int\limits_{v_{M_n}^{(n)}}^{u_0^{(n)}} y^2\big(\pi_X(y) - \pi_X^{(n)}(y)\big)^+\,dy, \tag{4.4.12}$$

while we prefer to stick with the diffusion coefficient of the original process $(X_t)_{t\geq 0}$ and rely instead on (4.4.3), (4.4.4), and $\sigma_n^2 := \sigma_X^2$. Since the difference between the two diffusion coefficients does not exceed ϵ_n, choosing one or the other approximation scheme may seem equivalent. However, aggregating small jumps into an additional diffusion factor transforms in particular infinite-activity pure jump processes into approximating hyper-exponential jump-diffusion processes with non-zero diffusion. When dealing with first-passage problems, this additional diffusion factor is known to artificially imply a smooth-pasting condition at the barrier level, which subsequently leads, near the barrier, to qualitative differences in the solutions to the first-passage problem under the original process and under the approximating processes (cf. [BL09], [BL12]). As we are particularly interested in quantifying intra-horizon risk for small α, we only need to compute first-passage probabilities for starting values far from the barrier. Consequently, relying on the same approach used in [AMP07] and [JP10] may still provide reasonable results (cf. [LV20]). Nevertheless, we prefer to follow a more natural approach and keep the pure jump structure of the original process by relying on (4.4.3), (4.4.4), and $\sigma_n^2 := \sigma_X^2$.

♦

4.4.2 Adaptions for Pure Jump Lévy Models

At this point, we have already emphasized that the structure of pure jump processes implies for the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^\pm}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^\pm}\big)(\cdot)$ that the continuous-pasting conditions

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^\pm}\big)(\vartheta, \ell\mp;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_0^\pm}\big)(\vartheta, \ell;\ell) \quad \text{and} \quad \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^\pm}\big)(\vartheta, \ell\mp;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^\pm}\big)(\vartheta, \ell;\ell) \tag{4.4.13}$$

do not anymore hold. Instead, when dealing with pure jump processes of infinite variation, these conditions need to be replaced by

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^\pm}\big)(\vartheta, \ell\mp;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_0^\pm}\big)(\vartheta, \ell\pm;\ell) \quad \text{and} \quad \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^\pm}\big)(\vartheta, \ell\mp;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^\pm}\big)(\vartheta, \ell\pm;\ell). \tag{4.4.14}$$

Although (4.4.14) is not anymore satisfied under hyper-exponential jump-diffusion approximations to pure jump processes of infinite variation, one may want to impose them anyway and analogous results to the ones in Proposition 4.4 can be derived under (4.4.14). Alternatively, jump contributions to (maturity-ramdomized) first-passage probabilities can be obtained by following the approach taken in Proposition 4.5 and subsequently solving the resulting systems of equations. This leads to the following analogue of Proposition 4.6. The reader is referred for a proof in a slightly different context to [CYY13].

Proposition 4.7. *Assume that $(X_t)_{t\geq 0}$ follows a hyper-exponential jump-diffusion process, as described in (4.3.43), (4.3.44) and with $\sigma_X = 0$ and $\mu \leq 0$. Then, for any level $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$ and intensity $\vartheta > 0$ we have that*

$$\mathcal{LC}\big(u_X^{\mathcal{E}_i^+}\big)(\vartheta, x;\ell) = \sum_{k=1}^{m} \tilde{v}_{\mathcal{E}_i^+,k} \cdot e^{\beta_{k,\vartheta}\cdot(x-\ell)}, \qquad i \in \{1,\ldots,m\}, \tag{4.4.15}$$

where the coefficients $\tilde{v}_{\mathcal{E}_i^+,k}$ are given by

$$\tilde{v}_{\mathcal{E}_i^+,k} = -\frac{\tilde{\mathbf{B}}^+(\beta_{k,\vartheta})\,\tilde{\mathbf{C}}_\vartheta^+(\xi_i)}{\xi_i\,(\xi_i - \beta_{k,\vartheta})\,\big(\tilde{\mathbf{B}}^+\big)'(\xi_i)\,\big(\tilde{\mathbf{C}}_\vartheta^+\big)'(\beta_{k,\vartheta})}, \qquad k \in \{1,\ldots,m\}, \tag{4.4.16}$$

with

$$\tilde{\mathbf{B}}^+(x) := \prod_{s=1}^{m} \big(\xi_s - x\big), \qquad \tilde{\mathbf{C}}_\vartheta^+(x) := \prod_{s=1}^{m} \big(\beta_{s,\vartheta} - x\big). \tag{4.4.17}$$

Similarly, if $\sigma_X = 0$ and $\mu \geq 0$, we obtain, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^-}$ and $\vartheta > 0$, that

$$\mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)(\vartheta, x;\ell) = \sum_{k=1}^{n} \tilde{v}_{\mathcal{E}_j^-,k} \cdot e^{\gamma_{k,\vartheta}\cdot(x-\ell)}, \qquad j \in \{1,\ldots,n\} \tag{4.4.18}$$

where the coefficients $\tilde{v}_{\mathcal{E}_j^-,k}$ *are given by*

$$\tilde{v}_{\mathcal{E}_j^-,k} = \frac{\tilde{\mathbf{B}}^-(\gamma_{k,\vartheta})\,\tilde{\mathbf{C}}_\vartheta^-(\eta_j)}{\eta_j\,(\eta_j+\gamma_{k,\vartheta})\,\big(\tilde{\mathbf{B}}^-\big)'(-\eta_j)\,\big(\tilde{\mathbf{C}}_\vartheta^-\big)'(-\gamma_{k,\vartheta})}, \qquad k \in \{1,\dots,n\}, \tag{4.4.19}$$

with

$$\tilde{\mathbf{B}}^-(x) := \prod_{s=1}^{n}\big(\eta_s + x\big), \qquad \tilde{\mathbf{C}}_\vartheta^-(x) := \prod_{s=1}^{n}\big(\gamma_{s,\vartheta} + x\big). \tag{4.4.20}$$

Remark 4.6.

i) We note that the coefficients $v_{\mathcal{E}_i^+,k}$, $\tilde{v}_{\mathcal{E}_i^+,k}$ and $v_{\mathcal{E}_j^-,k}$, $\tilde{v}_{\mathcal{E}_j^-,k}$ in Proposition 4.6 and Proposition 4.7 depend on the intensity $\vartheta > 0$ chosen in the Laplace-Carson transform. In order to bear this dependency in mind when dealing with applications of the Gaver-Stehfest inversion algorithm in the next section, we will sometimes express these coefficients as functions of $\vartheta > 0$ and write $v_{\mathcal{E}_i^+,k}(\vartheta)$, $\tilde{v}_{\mathcal{E}_i^+,k}(\vartheta)$, and $v_{\mathcal{E}_j^-,k}(\vartheta)$, $\tilde{v}_{\mathcal{E}_j^-,k}(\vartheta)$ instead.

ii) To conclude our analysis, we observe that, for $\sigma_X = 0$ and $\mu > 0$ (or $\sigma_X = 0$ and $\mu < 0$), the same results as in Proposition 4.6 hold, i.e. we have that, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$ (or $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^-}$) and $\vartheta > 0$, the functions $\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_i^+}\big)(\cdot)$, for $i \in \{1,\dots,m\}$, (or $\mathcal{LC}\big(u_X^{\mathcal{E}_0^-}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)(\cdot)$, for $j \in \{1,\dots,n\}$,) are given via (A.4.54) and (4.3.74), (4.3.75) (or (A.4.57) and (4.3.78), (4.3.79) respectively). This follows by combining the results in Lemma 4.2 with Proposition 4.5, since considering these two cases do not alter the number of positive (or negative) roots to the equation $\Phi_X(\theta) = \vartheta$ and, therefore, the system of equations (4.3.72) (or (4.3.73)). Further details can be also found in [CYY13].

♦

4.5 Calibration and Numerical Results

To illustrate the practicability of the intra-horizon risk measurement approach developed in the previous sections, we lastly analyze the 10-days intra-horizon risk inherent to a long position in the S&P 500 index over (approx.) 24 years. More specifically, we focus on the case where the (discounted) profit and loss process reflects the intrinsic value of a long position in the S&P 500 index (cf. *Scenario 2* in Section 4.3) and derive historical intra-horizon risk results by calibrating Variance-Gamma (VG) and Carr-Geman-Madan-Yor (CGMY) dynamics to S&P 500 index data and subsequently approximating the intra-horizon risk in these models by combining the general approach of Section 4.3 with the methods presented in Section 4.4.

4.5.1 Data

Our data set comprises historical returns of the S&P 500 index from January 1990 until April 2019, therefore spanning almost three decades. During this period, a wide variety of macroeconomic, financial, and political risk factors have influenced the performance of the US equity market. Following [BP10], we consider weekly frequency in our empirical study, which gives us 1,269 observations in the sample. As discussed in [LV20], weekly returns are suboptimal in the sense that there is a mismatch between the sampling frequency and the 10-days horizon that is typically considered in risk management applications. However, using biweekly returns would halve the number of observations, which would further exacerbate estimation problems regarding the risk measures considered in this paper. Therefore, similarly to the previous studies, our decision to rely on weekly returns represents a trade-off between the quality of our estimation results and the accuracy of the sampling frequency.

4.5.2 Calibration Method and Results

Our approach closely follows [BP10] – We estimate parameters of certain Lévy models on a rolling-window basis using a maximum likelihood estimation (MLE) procedure. The only difference is that we rely on the Fourier cosine method of [FO08] to estimate the probability density function of the weekly index returns. This method is very fast and has been already recommended for a similar application in [LV20].[8]

We consider two Lévy models that are well established in the literature and widely applied in practice: the Variance-Gamma (VG) model and the Carr-Geman-Madan-Yor (CGMY) model. In the case of the CGMY model, we set the fine structure parameter, Y, to $Y = 0.5$. This particular choice was proposed in [BP10], mainly for two reasons. First, the resulting model has an infinite-activity-and-finite-variation property which is known to describe time series of equity returns very well. Second, having this parameter fixed allows for a better identification of the remaining model parameters, i.e., the jump arrival rate C, and the exponential decay parameters G and M. For the VG model, we note that the resulting dynamics corresponds to a special case of the CGMY dynamics – Here, the fine structure parameter is given by $Y = 0$.

Table 4.1: Summary statistics for the calibrated parameters. We estimate parameters of the Variance-Gamma (VG) and the Carr-Geman-Madan-Yor (CGMY) models using weekly historical returns of the S&P 500 index from January 1990 until April 2019 (the total number of observations is 1,269) on a five-years rolling-window basis. The table reports the average and median values, the standard deviations and the mean absolute deviations (MAD) for the estimated parameters, as well as the negative log-likelihood (MLE). The parameters C, G and M are based on unconstrained calibrations. We set the values of the parameter Y to $Y = 0$ and $Y = 0.5$ for VG and CGMY, respectively.

Summary Statistics						
Models	*Parameters*	C	G	M	Y	*MLE*
VG	*Mean*	94.80	75.49	112.83	0	4.42
	Median	72.98	74.20	105.28	0	4.41
	Std. Dev.	47.43	24.19	47.09	–	0.23
	MAD	20.10	18.82	16.89	–	0.21
CGMY	*Mean*	6.76	49.33	85.22	0.5	4.42
	Median	5.39	46.10	75.90	0.5	4.41
	Std. Dev.	3.33	19.17	39.14	–	0.23
	MAD	1.30	15.77	18.22	–	0.22

Table 4.1 summarizes our calibration results for the two models under consideration. Besides reporting average values and standard deviations for all model parameters, we have chosen to include the median as well as the median absolute deviation. This is important as the mean may be influenced by outliers and the non-normality of the data.

[8] Details around the calibration are thoroughly discussed in the two cited papers, hence we do not elaborate here further on the estimation procedure.

Several patterns can be observed over time and across the estimates. First, the jump intensity parameter C was spiked during the 1997 Asian Financial Crisis and the 2008-2009 Global Financial Crisis. Interestingly, the jump intensity was rather high during the 2002-2007 Bull Market, however it was not accompanied by elevated positive and/or negative jumps. Second, the expected size of positive jumps – i.e. the inverse of the parameter G – is rather stable at the level of about 1-2% for the whole sample and tends to become smaller during the crisis periods. Third, the expected size of negative jumps – i.e. the inverse of the parameter M – was particularly high (around 3.5-4.0%) during the 2000 Dot-Com Bubble Burst and following the 2008-2009 Global Financial Crisis. This parameter exhibits high persistence and clustering behavior as it remains elevated long after a bear market is over. In recent years, this parameter has been increasing against the backdrop of slower global economic growth and increased political risks. Except for the 2002-2007 Bull Market, the left tail of the returns distribution was always heavier than the right tail, i.e. $G > M$. Last but not least, we stress that the key driver of the differences between the reported parameter estimates is the fine structure parameter Y which is fixed at $Y = 0$ and $Y = 0.5$ for VG and CGMY, respectively. Overall, we observe that the VG model exhibits a higher activity of small jumps and a more symmetric distribution of returns.

4.5.3 Empirical Intra-Horizon Risk Results

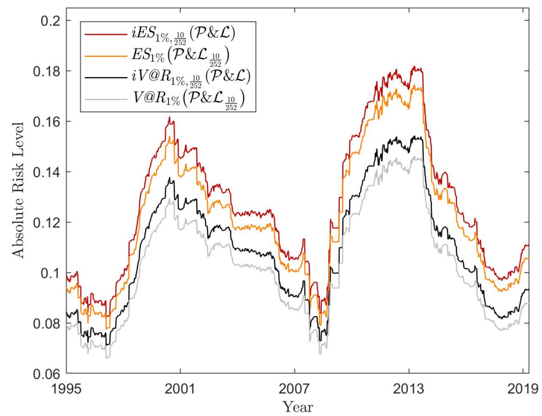
Having calibrated the VG and CGMY dynamics to S&P 500 index data ranging from January 1990 to April 2019, we next turn to a derivation of the 10-days intra-horizon risk in these models. We compute intra-horizon risk results based on hyper-exponential jump-diffusion approximations and the combination of the theory developed in Section 4.3, Proposition 4.7, and the Gaver-Stehfest inversion algorithm. More specifically, we follow the ideas of [AMP07] (cf. also [LV20]), i.e. we fix in advance the number of exponentials N_n and M_n in the approximating density (4.4.3), (4.4.4) and minimize the distance between the approximating and the true Lévy densities by optimally choosing the partition of the integration intervals. This slightly differs from the approach used in [JP10], where the authors additionally fix all mean jump sizes $(\xi_i)^{-1}$ and $(\eta_j)^{-1}$ and subsequently use least-squares optimization to determine the values of the remaining (mixing) parameters. However, while these authors only work with few exponentials,[9] we choose $N_n = 100$ and $M_n = 100$ and incorporate this way 200 exponentials. This additionally ensures that we approximate small jumps sufficiently well, as we have decided to keep the pure jump structure of the approximated processes by refraining from converting small jumps into an extra diffusion factor (cf. Remark 4.5. in Section 4.4).[10]

As soon as hyper-exponential jump-diffusion approximations are fixed, we make use of the derivations in the previous sections to derive intra-horizon risk results in the following way: First, 10-days intra-horizon value-at-risk measures as well as corresponding risk contributions per jump type are obtained by combining Proposition 4.7 with Relations (4.2.5), (4.3.6) and (4.3.26), i.e. we invert the functions $\mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)(\cdot)$, for any $j \in \{1, \ldots, n\}$, via

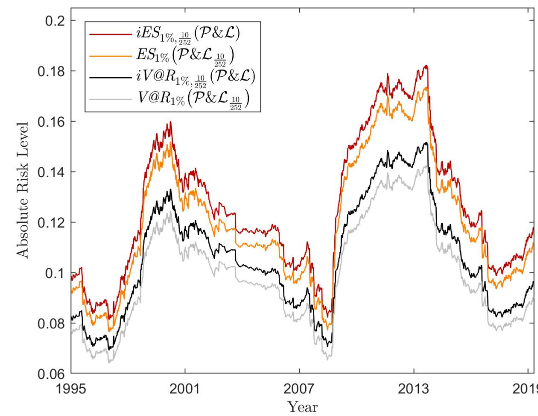
$$u_X^{\mathcal{E}_j^-}(T, x; \ell) = \lim_{N \to \infty} \left(u_X^{\mathcal{E}_j^-}\right)_N (T, x; \ell),$$

[9] The authors in [JP10] only use a total of 14 exponentials, i.e. 7 exponentials for both the positive and the negative parts of the distribution.

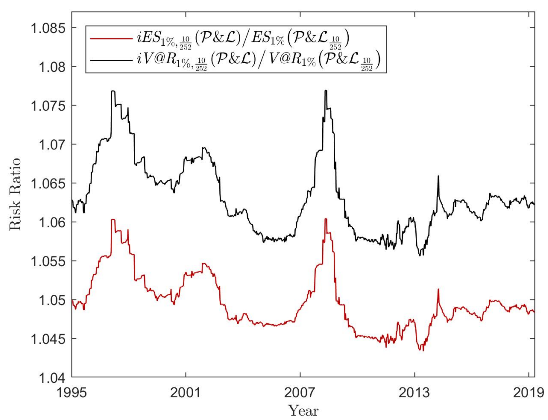
[10] Our numerical tests show that the suggested procedure is fast and stable. However, we emphasize that other approaches exist in the literature (cf. for instance [CLM10]) and that investigating the performance of all these algorithms is not the sake of this article, but constitutes a separate research topic.



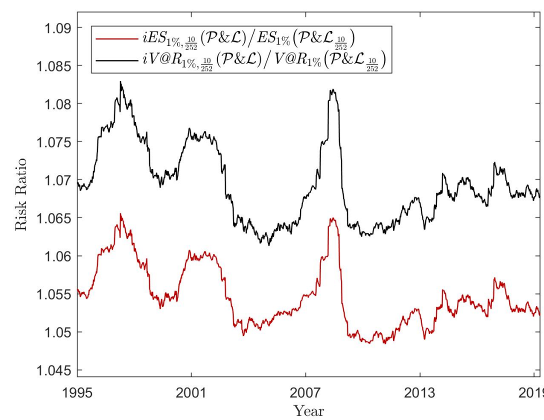
(a) Evolution of Absolute Risk (VG)



(b) Evolution of Absolute Risk (CGMY)



(c) Intra-Horizon to Point-in-Time Risk Ratio (VG)



(d) Intra-Horizon to Point-in-Time Risk Ratio (CGMY)

Figure 4.1: Comparison of the intra-horizon and point-in-time risk for the S&P 500 index. Figure 4.1a and Figure 4.1b show the time evolution of the 10-days intra-horizon risk $\big($intra-horizon value at risk, $iV@R_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})$, and intra-horizon expected shortfall, $iES_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})\big)$ and 10-days point-in-time risk $\big($(standard) value at risk, $V@R_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)$, and (standard) expected shortfall, $ES_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)\big)$ to the 99% loss quantile from January 1995 until April 2019. The resulting absolute risk levels correspond to negative return levels under the respective dynamics. Additionally, Figure 4.1c and Figure 4.1d provide the risk ratio of intra-horizon risk to point-in-time risk under the respective Lévy models.

$$\begin{aligned}\left(u_X^{\mathcal{E}_j^-}\right)_N(T,x;\ell) &:= \sum_{k=1}^{2N} \zeta_{k,N}\,\mathcal{LC}\big(u_X^{\mathcal{E}_j^-}\big)\Big(\frac{k\log(2)}{T},x;\ell\Big)\\ &= \sum_{k=1}^{2N}\sum_{s=1}^{n} \zeta_{k,N}\,\tilde{v}_{\mathcal{E}_j^-,s}\Big(\frac{k\log(2)}{T}\Big)\,\exp\Big\{\gamma_{s,\frac{k\log(2)}{T}}\cdot(x-\ell)\Big\},\end{aligned} \tag{4.5.1}$$

with $\zeta_{k,N}$ defined as in (4.3.25), and derive the respective results based on Relations (4.2.5), (4.3.6) and the ideas introduced in Section 4.3.3. Once these quantities are obtained, recovering 10-days intra-horizon expected shortfall results reduces to the evaluation of integrals and of fractions of integrals of the form of (4.3.38) and (4.3.42). Here, combining (4.5.3), (4.5.1) with the monotonicity of the function $\ell \mapsto u_X^{\mathcal{E}_j^-}(T,x;\ell)$ allows us to derive, for each $j\in\{1,\ldots,n\}$, that

$$\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} u_X^{\mathcal{E}_j^-}\big(T,\log(z_2+z);\log(z_2+\ell)\big)\,d\ell = \lim_{N\to\infty}\left(\mathcal{I}_{\alpha,T}^{\mathcal{E}_j^-}\right)_N(z,z_2), \tag{4.5.2}$$

where $\left(\mathcal{I}_{\alpha,T}^{\mathcal{E}_j^-}\right)_N(z,z_2)$ is given, for any $N\in\mathbb{N}$, via

$$\begin{aligned}\left(\mathcal{I}_{\alpha,T}^{\mathcal{E}_j^-}\right)_N(z,z_2) &:= \int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})} \left(u_X^{\mathcal{E}_j^-}\right)_N\big(T,\log(z_2+z);\log(z_2+\ell)\big)\,d\ell\\ &= \sum_{k=1}^{2N}\sum_{s=1}^{n}\zeta_{k,N}\,\tilde{v}_{\mathcal{E}_j^-,s}\Big(\frac{k\log(2)}{T}\Big)\int_{-z_2}^{-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})}\left(\frac{z_2+z}{z_2+\ell}\right)^{\gamma_{s,\frac{k\log(2)}{T}}}d\ell\\ &= \sum_{k=1}^{2N}\sum_{s=1}^{n}\zeta_{k,N}\,\tilde{v}_{\mathcal{E}_j^-,s}\Big(\frac{k\log(2)}{T}\Big)\frac{z_2+z}{1-\gamma_{s,\frac{k\log(2)}{T}}}\left(\frac{z_2-iV@R_{\alpha,T}(\mathcal{P}\&\mathcal{L})}{z_2+z}\right)^{1-\gamma_{s,\frac{k\log(2)}{T}}}.\end{aligned} \tag{4.5.3}$$

This finally provides us with a simple numerical scheme to compute 10-days intra-horizon expected shortfall measures as well as, based on the ideas outlined in Section 4.3.3, corresponding risk contributions per jump type inherent to any long position in the S&P 500 index.

4.5.3.1 Intra-Horizon vs. Point-in-Time Risk

We now turn to the empirical risk results and start by providing a comparison of the intra-horizon and point-in-time risks inherent to a long position in the S&P 500 index from January 1995 to April 2019. To this end, we have plotted in Figure 4.1a and Figure 4.1b the time evolution of the absolute 10-days intra-horizon and point-in-time risks to the 99% quantile of the loss distribution[11] calculated under the respective Lévy dynamics. These results express intra-horizon and point-in-time risks in terms of (negative) return levels, i.e. the graphs were obtained by computing the respective risk measures while fixing $z_1=z_2=1$ in *Scenario 2* (cf. Section 4.3). To complement these results, we have also provided in Figure 4.1c and Figure 4.1d the time evolution of the intra-horizon to point-in-time risk ratio. Finally, Figure 4.2 presents the evolution of the intra-horizon and point-in-time ratios $\omega_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L}) := iV@R_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})/iES_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})$ and $\omega_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big) := V@R_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)/ES_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)$ for both model dynamics. While we have chosen to

[11] In our notation, this corresponds to fixing $\alpha = 1\%$.

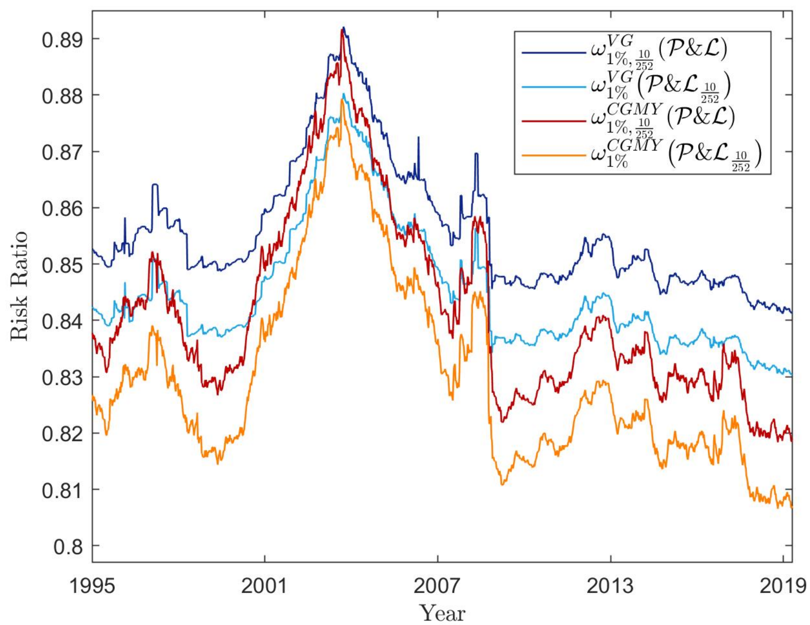
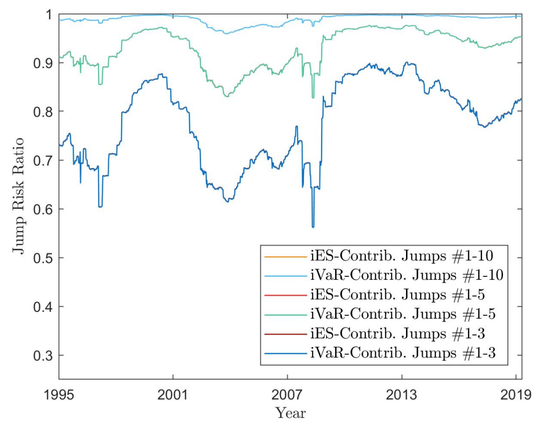


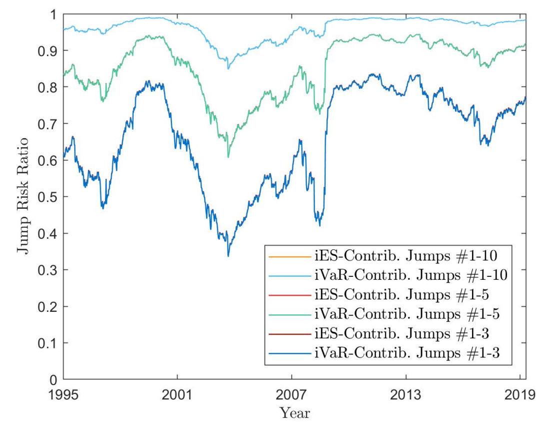
Figure 4.2: Time evolution of the intra-horizon/point-in-time value at risk to expected shortfall ratios. We have plotted for both VG and CGMY the time evolution of the intra-horizon and point-in-time risk ratios $\omega_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L}) := iV@R_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})/iES_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})$ and $\omega_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big) := V@R_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)/ES_{1\%}\big(\mathcal{P}\&\mathcal{L}_{\frac{10}{252}}\big)$, respectively. These ratios give the relative contribution of the 10-days intra-horizon/point-in-time value at risk to the 10-days intra-horizon/point-in-time expected shortfall to the 99% quantile of the loss distribution under the respective Lévy dynamics.

follow the framework of the Basel Accords (cf. [BCBS19]) and to provide results for a 10-days horizon, we note that we do not rely on the 97.5% quantile of the loss distribution prescribed in [BCBS19], but prefer to investigate the 99% level. This is mainly to stay consistent with the existing literature on intra-horizon risk quantification (cf. [BRSW04], [Ro08], [BMK09], [BP10], [LV20]) and to allow for a direct comparability of our results with other articles.

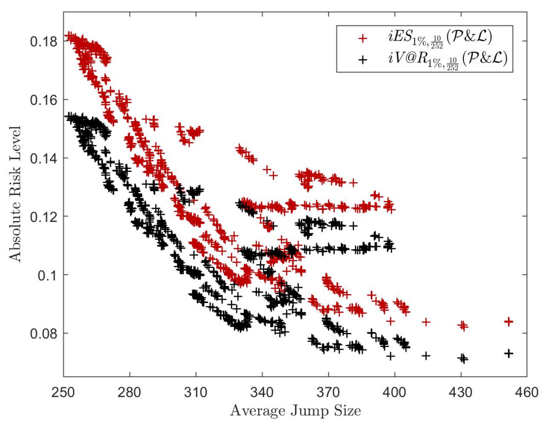
The results in Figure 4.1 and Figure 4.2 are in line with our intuition: First, the 10-days intra-horizon expected shortfall to the 99% loss quantile, $iES_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})$, exceeds at any time the intra-horizon value at risk at the same level, $iV@R_{1\%,\frac{10}{252}}(\mathcal{P}\&\mathcal{L})$, and the same additionally holds true for the point-in-time measures. Moreover, intra-horizon risk measures always exceed their point-in-time equivalent. This becomes evident when looking at Figures 4.1a-4.1d where the intra-horizon risk curve is always higher than its point-in-time reference and the intra-horizon to point-in-time risk ratio never falls below 1.0. In particular, Figure 4.1c and Figure 4.1d show that this ratio has a similar structure for both (intra-horizon) value at risk and (intra-horizon) expected shortfall, however, that it is greater for the (intra-horizon) value at risk. Finally, we note that intra-horizon risk is generally 5-8% higher than point-in-time risk. Next, when investigating any of Figure 4.1 and Figure 4.2, one sees that all the intra-horizon and point-in-time measures behave similarly. In particular, all the lines in Figure 4.1a and Figure 4.1b exhibit an (almost) identical shape and seem to be obtained via a parallel shift of anyone of them. However, a closer look at these graphs reveals that the absolute difference between intra-horizon/point-in-time expected shortfall and intra-horizon/point-in-time



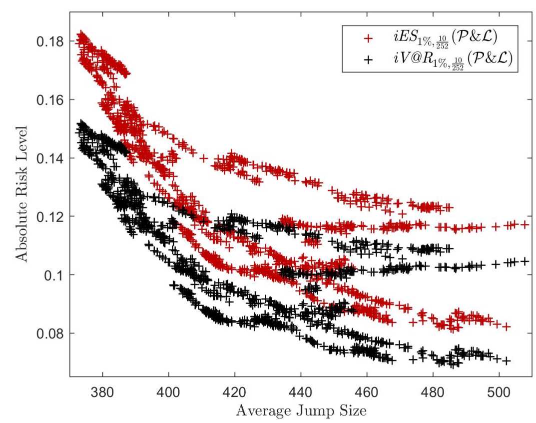
(a) Evolution of Jump Contributions (VG)



(b) Evolution of Jump Contributions (CGMY)



(c) Average Jump Size and Absolute Risk (VG)



(d) Average Jump Size and Absolute Risk (CGMY)

Figure 4.3: Comparison of the intra-horizon risk and the contribution of certain jumps. Figure 4.3a and Figure 4.3b show the time evolution of the 10-days intra-horizon risk contributions to the 99% loss quantile for the greatest – in absolute size – 3 jumps, the greatest 5 jumps, and the greatest 10 jumps in the hyper-exponential jump-diffusion approximations. Additionally, Figure 4.3c and Figure 4.3d present the relation of the (absolute) average jump size – weighted by the probability of occurrence of each jump in the hyper-exponential jump-diffusion approximations – to the absolute intra-horizon risk level. As earlier, the absolute risk levels correspond to negative return levels under the respective dynamics.

value at risk tends to substantially increase in more severe times. That this behavior does not only hold at an absolute level but also in relative terms can be seen in Figure 4.2 where the intra-horizon/point-in-time value-at-risk contribution to the intra-horizon/point-in-time expected shortfall tends to take lower values in crisis periods.

4.5.3.2 Intra-Horizon Risk and Structure of Risk Across Jumps

To finalize our discussion, we investigate the structure of intra-horizon risk across jumps.[12] To this end, we present in Figure 4.3 a comparison of intra-horizon risk and jump contributions. In particular, we have plotted in Figure 4.3a and Figure 4.3b the time evolution of the 10-days intra-horizon risk contributions to the 99% loss quantile for the greatest – in absolute size – 3 jumps, the greatest 5 jumps, and the greatest 10 jumps in the hyper-exponential jump-diffusion approximations. Additionally, Figure 4.3c and Figure 4.3d show the relation of the (absolute) average jump size – weighted by the probability of occurrence of each jump – to the absolute intra-horizon risk level. The results are in line with the existing literature (cf. e.g. [LV20]) as well as with the observations in the previous section. First, we note that the greatest 3, 5, and 10 jumps in the hyper-exponential jump-diffusion approximations already provide a high contribution to both intra-horizon value at risk and intra-horizon expected shortfall – The first 3, 5, and 10 jumps have a slightly higher contribution for VG than for CGMY, with roughly 55-85%, 80-95%, and 95-99% for VG compared to 40-75%, 70-90%, and 90-97% for CGMY. Additionally, looking at Figures 4.3a-4.3d reveals that the structure of risk across jumps does not differ for both of these risk measures. Indeed, while the risk contributions per jump types to both intra-horizon value at risk and intra-horizon expected shortfall are almost identical, the intra-horizon expected shortfall results in Figure 4.3c and Figure 4.3d merely replicate the shape of the intra-horizon value-at-risk results at a slightly higher risk level. This is due to the fact that the intra-horizon expected shortfall always exceeds the intra-horizon value at risk for the same time horizon and quantile. Lastly, we emphasize that Figure 4.3c and Figure 4.3d present evidence of the fact that higher (absolute) average jumps generally lead to higher absolute risk levels. This is intuitively clear, since greater (absolute) average jumps immediately increase the tail of the jump distribution which likewise impacts the tail of the overall profit and loss distribution.

4.6 Conclusion

The present article extended the current literature on intra-horizon risk quantification in several directions. First, we proposed an intra-horizon analogue of the expected shortfall and discussed some of its key properties under general Lévy dynamics. The resulting (intra-horizon) risk measure is well-defined for (m)any popular class(es) of Lévy processes encountered in financial modeling and constitutes a coherent measure of risk in the sense of [CDK04]. Secondly, we linked our intra-horizon expected shortfall to first-passage occurrences and derived a characterization of diffusion and jump contributions to simple and maturity-randomized first-passage probabilities. These results were subsequently used to infer diffusion and jump risk contributions to the intra-horizon expected shortfall and additionally allowed us to obtain (semi-)analytical results for maturity-randomized first-passage probabilities under hyper-exponential jump-diffusion dynamics. Next, we reviewed hyper-exponential jump-diffusion approximations to Lévy processes having completely monotone jumps and proposed an adaption of the results in [AMP07], [JP10] that naturally preserves the diffusion vs. jump structure of the approximated processes. We then calibrated popular (pure jump) Lévy processes to S&P 500 index data and combined several of our results to analyze the intra-horizon risk inherent to a long position in the S&P 500 index from January 1995 to April 2019. Our empirical findings revealed that

[12] We emphasize that similar results can be derived for point-in-time risk. However, due to the focus of the paper, we only provide intra-horizon risk results.

even when considering large loss quantiles (i.e. low α) the intra-horizon value at risk and the intra-horizon expected shortfall add conservatism to their point-in-time estimates. Additionally, they suggested that these risk measures have a very similar structure across jumps/jump clusters and that already a high contribution of their risk is due to only few, great – in terms of the absolute jump size – jump clusters.

4.7 Appendices

4.7.1 Appendix A: Proofs – General Results

Proof of Proposition 4.1. To start, we note that Proposition 3.2 in [AT02] implies that for any $T > 0$ and $\alpha \in (0,1)$ the intra-horizon expected shortfall associated to the profit and loss process $(\mathcal{P}\&\mathcal{L}_t)_{t\in[0,T]}$, $iES_{\alpha,T}(\mathcal{P}\&\mathcal{L})$, is given by

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) = -\frac{1}{\alpha}\bigg(\mathbb{E}_z^{\mathbb{P}}\Big[I_T^{\mathcal{P}\&\mathcal{L}}\,\mathbb{1}_{\{I_T^{\mathcal{P}\&\mathcal{L}}\leq q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})\}}\Big] - q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})\,\Big[\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big) - \alpha\Big]\bigg). \tag{A.4.1}$$

Therefore, we next derive an expression for the integral/expectation part in (A.4.1) and will subsequently use the result to recover (4.2.11). Here, noting that, under $\mathbb{P}_z$, the inequality $I_T^{\mathcal{P}\&\mathcal{L}} \leq z$ holds for any $T \geq 0$ allows us to write

$$\begin{aligned}
\mathbb{E}_z^{\mathbb{P}}\Big[I_T^{\mathcal{P}\&\mathcal{L}}\,\mathbb{1}_{\{I_T^{\mathcal{P}\&\mathcal{L}}\leq q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})\}}\Big] &= \mathbb{E}_z^{\mathbb{P}}\Big[-\int_{I_T^{\mathcal{P}\&\mathcal{L}}}^{z} \mathbb{1}_{\{I_T^{\mathcal{P}\&\mathcal{L}}\leq q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})\}}\,d\ell\Big] + z\,\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big) \\
&= -\int_{-\infty}^{z} \mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq \ell,\, I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big)\,d\ell + z\,\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big) \\
&= -\int_{q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})}^{z} \mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big)\,d\ell - \int_{-\infty}^{q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})} \mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq \ell\Big)\,d\ell \\
&\qquad + z\,\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big) \\
&= q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\,\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big)\Big) - \int_{-\infty}^{q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})} \mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq \ell\Big)\,d\ell.
\end{aligned} \tag{A.4.2}$$

Hence, combining (A.4.1) and (A.4.2) with the relation $\mathbb{P}_z\Big(I_T^{\mathcal{P}\&\mathcal{L}} \leq \ell\Big) = \mathbb{P}_z\Big(\tau_\ell^{\mathcal{P}\&\mathcal{L},-} \leq T\Big)$ gives that the intra-horizon expected shortfall can be expressed in terms of first-passage probabilities, as

$$iES_{\alpha,T}(\mathcal{P}\&\mathcal{L}) = \frac{1}{\alpha}\int_{-\infty}^{q_\alpha(I_T^{\mathcal{P}\&\mathcal{L}})} \mathbb{P}_z\Big(\tau_\ell^{\mathcal{P}\&\mathcal{L},-} \leq T\Big)\,d\ell - q_\alpha\big(I_T^{\mathcal{P}\&\mathcal{L}}\big), \tag{A.4.3}$$

which finally provides Equation (4.2.11). □

Proof of Lemma 4.1. To show (4.3.8), we rely on similar arguments to the ones used in [CK11] (cf. also [HM13]). Here, we focus on the result for the upside first-passage probabilities and show that there exists a constant $c > 1$ such that for any $x \in \mathbb{R}$ and $\mathcal{T} > 0$ we have that

$$\lim_{\ell\uparrow\infty} e^{c\cdot\ell}\,\mathbb{P}_x^X\Big(M_{\mathcal{T}}^X \geq \ell\Big) = 0, \tag{A.4.4}$$

where $(M_t^X)_{t\geq 0}$ denotes the maximum process associated to $(X_t)_{t\geq 0}$, i.e. the process defined by

$$M_t^X := \sup_{0\leq u\leq t} X_u, \qquad t \geq 0. \tag{A.4.5}$$

Once (A.4.4) is established, the respective convergence result for the downside first-passage probabilities is easily obtained by noting that the minimum process $(I_t^X)_{t\geq 0}$ satisfies

$$M_t^{\tilde{X}} = -I_t^X, \qquad t \geq 0, \tag{A.4.6}$$

where we have denoted by $(\tilde{X}_t)_{t\geq 0}$ the dual process to $(X_t)_{t\geq 0}$, i.e. the process that is defined by $\tilde{X}_t := -X_t$, $t \geq 0$. Therefore, we only have to prove (A.4.4). Here, we start by recalling that the process $(Z_t^\theta)_{t\geq 0}$ defined via

$$Z_t^\theta := e^{\theta X_t - t\Phi_X(\theta)}, \qquad t \geq 0, \tag{A.4.7}$$

is, for any $\theta \in \mathbb{R}$ satisfying $\mathbb{E}_0^{\mathbb{P}^X}\left[e^{\theta X_1}\right] < \infty$, a well-defined martingale. Using the optional sampling theorem, this allows us to derive, in particular, that for $\theta^\star > 1$ and any $x \in \mathbb{R}$, $\mathcal{T} > 0$ we have that

$$e^{\theta^\star \ell}\,\mathbb{P}_x^X\left(M_{\mathcal{T}}^X \geq \ell\right) \leq \mathbb{E}_x^{\mathbb{P}^X}\left[\exp\left\{\theta^\star X_{(\tau_\ell^{X,+}\wedge\mathcal{T})}\right\}\right] \leq \begin{cases} e^{\Phi_X(\theta^\star)\mathcal{T}+\theta^\star x}, & \text{if } \Phi_X(\theta^\star) > 0, \\ e^{\theta^\star x}, & \text{if } \Phi_X(\theta^\star) \leq 0. \end{cases} \tag{A.4.8}$$

Therefore, we obtain in any case for $x \in \mathbb{R}$ and $\mathcal{T} > 0$ that $e^{\theta^\star \ell}\,\mathbb{P}_x^X\left(M_{\mathcal{T}}^X \geq \ell\right) \leq C$ for some constant $C > 0$ and combining this result with the fact that $\theta^\star > 1$ finally gives, with $\theta_0 > 1$ and $c > 1$ satisfying $c\,\theta_0 = \theta^\star$, that for any $x \in \mathbb{R}$ and $\mathcal{T} > 0$

$$e^{c\cdot\ell}\,\mathbb{P}_x^X\left(M_{\mathcal{T}}^X \geq \ell\right) = e^{c(1-\theta_0)\ell}\,e^{c\,\theta_0\cdot\ell}\,\mathbb{P}_x^X\left(M_{\mathcal{T}}^X \geq \ell\right) = e^{c(1-\theta_0)\ell}\,e^{\theta^\star\ell}\,\mathbb{P}_x^X\left(M_{\mathcal{T}}^X \geq \ell\right) \to 0, \quad \text{as } \ell \uparrow \infty, \tag{A.4.9}$$

holds, hence (A.4.4).

To prove that $\mathbb{E}_z^{\mathbb{P}}\left[|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\right] < \infty$ holds for any $\mathcal{T} > 0$, we combine the convergence results (4.3.8) with standard techniques. First, we note that

$$\begin{aligned} \mathbb{E}_z^{\mathbb{P}}\left[|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\right] &= \mathbb{E}_z^{\mathbb{P}}\left[\int_0^\infty \mathbb{1}_{\{|I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}}|\geq\ell\}}\,d\ell\right] \\ &\leq \int_0^\infty \mathbb{P}_z\left(I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \geq \ell\right) d\ell + \int_0^\infty \mathbb{P}_z\left(I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \leq -\ell\right) d\ell \\ &\leq |z| + \int_0^\infty \mathbb{P}_z\left(I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \leq -\ell\right) d\ell. \end{aligned} \tag{A.4.10}$$

Therefore, we only need to show the finiteness of the integral on the right hand side. Under *Scenario 1*, the boundedness of $\ell \mapsto e^{c\cdot\ell}\,\mathbb{P}_z^X\left(\tau_{-\ell}^{X,-} \leq \mathcal{T}\right)$ on $[0,\infty)$ implies that

$$\int_0^\infty \mathbb{P}_z\left(I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \leq -\ell\right) d\ell = \int_0^\infty e^{-c\cdot\ell}e^{c\cdot\ell}\,\mathbb{P}_z^X\left(\tau_{-\ell}^{X,-} \leq \mathcal{T}\right) d\ell \leq K_1 \int_0^\infty e^{-c\cdot\ell}\,d\ell < \infty, \tag{A.4.11}$$

and this already provides the required result. Therefore, we next focus on *Scenario 2*. Here, we first note that for a long position the finiteness of the integral directly follows from the fact that $I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \geq -z_2$. Hence, we are left with the case of a short position under *Scenario 2*. In this case, similar arguments as in (A.4.11) give that

$$\int_0^\infty \mathbb{P}_z\Big(I_{\mathcal{T}}^{\mathcal{P}\&\mathcal{L}} \leq -\ell\Big)\, d\ell = \int_0^\infty e^{-c\log(z_2+\ell)} e^{c\log(z_2+\ell)}\, \mathbb{P}^X_{\log(z_2-z)}\Big(\tau^{X,+}_{\log(z_2+\ell)} \leq \mathcal{T}\Big)\, d\ell$$
$$\leq K_2 \left(1 + \int_1^\infty \frac{1}{(z_2+\ell)^c}\, d\ell\right) < \infty, \tag{A.4.12}$$

where the finiteness follows since $c > 1$. This finally gives the claim. □

4.7.2 Appendix B: Proofs – First-Passage Probabilities

Proof of Proposition 4.2. We start by noting that, for any $(\mathbf{t}, x, \delta) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, the process $(Z_t)_{t\in[0,\mathbf{t}]}$ defined via $Z_t := (\mathbf{t} - t, x + X_t, \delta + \Delta X_t)$ is a strong Markov process with state domain given by $\mathcal{D}_{\mathbf{t}} := [0, \mathbf{t}] \times \mathbb{R} \times \mathbb{R}$ and define, for any $\ell \in \mathbb{R}$, the following stopping domains

$$\begin{aligned}
&\mathcal{S}^+_\ell := \mathcal{S}^+_{\ell,1} \cup \mathcal{S}^+_{\ell,2}, \quad \text{with} \quad \mathcal{S}^+_{\ell,1} := \{0\}\times \mathbb{R}\times\mathbb{R} \quad \text{and} \quad \mathcal{S}^+_{\ell,2} := [0,T]\times[\ell,\infty)\times[\delta,\infty),\\
&\mathcal{S}^{\mathcal{J},+}_\ell := [0,T]\times(\ell,\infty)\times[\delta,\infty), \qquad \mathcal{S}^{0,+}_\ell := \mathcal{S}^+_\ell \setminus \mathcal{S}^{\mathcal{J},+}_\ell = \mathcal{S}^{0,+}_{\ell,1} \cup \mathcal{S}^{0,+}_{\ell,2},\\
&\text{with} \quad \mathcal{S}^{0,+}_{\ell,1} = \mathcal{S}^+_{\ell,1} \setminus \mathcal{S}^{\mathcal{J},+}_\ell \quad \text{and} \quad \mathcal{S}^{0,+}_{\ell,2} = \mathcal{S}^+_{\ell,2} \setminus \mathcal{S}^{\mathcal{J},+}_\ell,
\end{aligned} \tag{A.4.13}$$

$$\begin{aligned}
&\mathcal{S}^-_\ell := \mathcal{S}^-_{\ell,1} \cup \mathcal{S}^-_{\ell,2}, \quad \text{with} \quad \mathcal{S}^-_{\ell,1} := \{0\}\times \mathbb{R}\times\mathbb{R} \quad \text{and} \quad \mathcal{S}^-_{\ell,2} := [0,T]\times(-\infty,\ell]\times(-\infty,\delta],\\
&\mathcal{S}^{\mathcal{J},-}_\ell := [0,T]\times(-\infty,\ell)\times(-\infty,\delta], \qquad \mathcal{S}^{0,-}_\ell := \mathcal{S}^-_\ell \setminus \mathcal{S}^{\mathcal{J},-}_\ell = \mathcal{S}^{0,-}_{\ell,1} \cup \mathcal{S}^{0,-}_{\ell,2},\\
&\text{with} \quad \mathcal{S}^{0,-}_{\ell,1} = \mathcal{S}^-_{\ell,1} \setminus \mathcal{S}^{\mathcal{J},-}_\ell \quad \text{and} \quad \mathcal{S}^{0,-}_{\ell,2} = \mathcal{S}^-_{\ell,2} \setminus \mathcal{S}^{\mathcal{J},-}_\ell.
\end{aligned} \tag{A.4.14}$$

Clearly, both $\mathcal{S}^+_\ell$ and $\mathcal{S}^-_\ell$ are closed in the state space $\mathcal{D}_T$. We therefore obtain that, for each of theses domains, the first entry times $\tau_{\mathcal{S}^\pm_\ell}$ defined via

$$\tau_{\mathcal{S}^\pm_\ell} := \inf\left\{t \geq 0 : \ Z_t \in \mathcal{S}^\pm_\ell\right\} \tag{A.4.15}$$

is a stopping time that satisfies $\tau_{\mathcal{S}^\pm_\ell} \leq \mathbf{t}$, under $\mathbb{P}^Z_z$, the measure having initial distribution $Z_0 = z = (\mathbf{t}, x, \delta)$. Using this notation, we can now re-express the first-passage probabilities $u_X^{\mathcal{E}_0^\pm}(\cdot)$ and $u_X^{\mathcal{E}_{\mathcal{J}}^\pm}(\cdot)$ as solutions of appropriate stopping problems. Indeed, it is easily seen that

$$u_X^{\mathcal{E}_0^\pm}(\mathcal{T}, x; \ell) = V_0^\pm\big((\mathcal{T}, x, 0)\big) \qquad \text{and} \qquad u_X^{\mathcal{E}_{\mathcal{J}}^\pm}(\mathcal{T}, x; \ell) = V_{\mathcal{J}}^\pm\big((\mathcal{T}, x, 0)\big), \tag{A.4.16}$$

where the value functions $V_0^\pm(\cdot)$ and $V_{\mathcal{J}}^\pm(\cdot)$ have the following probabilistic representations:

$$V_0^\pm(z) = \mathbb{E}_z^{\mathbb{P}^Z}\left[\mathbb{1}_{\mathcal{S}^{0,\pm}_{\ell,2}}\Big(Z_{\tau_{\mathcal{S}^\pm_\ell}}\Big)\right] \qquad \text{and} \qquad V_{\mathcal{J}}^\pm(z) = \mathbb{E}_z^{\mathbb{P}^Z}\left[\mathbb{1}_{\mathcal{S}^{\mathcal{J},\pm}_\ell}\Big(Z_{\tau_{\mathcal{S}^\pm_\ell}}\Big)\right]. \tag{A.4.17}$$

Additionally, standard arguments based on the strong Markov property (cf. [PS06], [Ma19], [Ma20]) imply that, for any $\ell \in \mathbb{R}$, the functions $V_0^\pm(\cdot)$ and $V_{\mathcal{J}}^\pm(\cdot)$ satisfy[13]

$$\partial_{\mathbf{t}} V_0^\pm\big((\mathbf{t}, x, \delta)\big) = \mathcal{A}_X V_0^\pm\big((\mathbf{t}, x, \delta)\big), \quad \text{on } \mathcal{D}_T \setminus \mathcal{S}^\pm_\ell, \tag{A.4.18}$$
$$V_0^\pm\big((\mathbf{t}, x, \delta)\big) = \mathbb{1}_{\mathcal{S}^{0,\pm}_{\ell,2}}\big((\mathbf{t}, x, \delta)\big), \quad \text{on } \mathcal{S}^\pm_\ell, \tag{A.4.19}$$

[13] Note that we implicitly use the fact that the infinitesimal generator of the process $(\Delta X_t)_{t\in[0,\mathbf{t}]}$ vanishes. This follows since $(X_t)_{t\in[0,\mathbf{t}]}$ is, as Feller process, quasi-left-continuous.

and

$$\partial_{\mathbf{t}} V_{\mathcal{J}}^{\pm}\big((\mathbf{t},x,\delta)\big) = \mathcal{A}_X V_{\mathcal{J}}^{\pm}\big((\mathbf{t},x,\delta)\big), \qquad \text{on } \mathcal{D}_T \setminus \mathcal{S}_\ell^{\pm}, \tag{A.4.20}$$

$$V_{\mathcal{J}}^{\pm}\big((\mathbf{t},x,\delta)\big) = \mathbb{1}_{\mathcal{S}_\ell^{\mathcal{J},\pm}}\big((\mathbf{t},x,\delta)\big), \qquad \text{on } \mathcal{S}_\ell^{\pm}. \tag{A.4.21}$$

Therefore, recovering $u_X^{\mathcal{E}_0^{\pm}}(\cdot)$ and $u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}(\cdot)$ via (A.4.16) directly gives Equations (4.3.14)-(4.3.16) and (4.3.18)-(4.3.20), respectively. Since Equations (4.3.17) and (4.3.21) are naturally satisfied, Proposition 4.2 follows. □

Proof of Proposition 4.3. We prove Proposition 4.3 using a similar approach to the one adopted in the proof of Proposition 4.2. First, we recall from (4.3.29) that any of the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$ can be seen as the probability of a respective first-passage event occurring before the first jump time of an independent Poisson process $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$. Therefore, we consider, for any $(n,x,\delta) \in \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R}$, the process $(Z_t)_{t\geq 0}$ defined via $Z_t := (n+N_t, x+X_t, \delta+\Delta X_t)$ and note that it is a strong Markov process on the state domain $\mathcal{D} := \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R}$. Additionally, we define, for $\ell \in \mathbb{R}$, the following stopping domains

$$\begin{gathered}
\mathcal{S}_\ell^{+} := \mathcal{S}_{\ell,1}^{+} \cup \mathcal{S}_{\ell,2}^{+}, \quad \text{with} \quad \mathcal{S}_{\ell,1}^{+} := \mathbb{N}\times\mathbb{R}\times\mathbb{R} \quad \text{and} \quad \mathcal{S}_{\ell,2}^{+} := \{0\}\times[\ell,\infty)\times[\delta,\infty), \\
\mathcal{S}_\ell^{\mathcal{J},+} := \{0\}\times(\ell,\infty)\times[\delta,\infty), \qquad \mathcal{S}_\ell^{0,+} := \mathcal{S}_\ell^{+}\setminus\mathcal{S}_\ell^{\mathcal{J},+} = \mathcal{S}_{\ell,1}^{0,+} \cup \mathcal{S}_{\ell,2}^{0,+}, \\
\text{with} \quad \mathcal{S}_{\ell,1}^{0,+} = \mathcal{S}_{\ell,1}^{+}\setminus\mathcal{S}_\ell^{\mathcal{J},+} \quad \text{and} \quad \mathcal{S}_{\ell,2}^{0,+} = \mathcal{S}_{\ell,2}^{+}\setminus\mathcal{S}_\ell^{\mathcal{J},+},
\end{gathered} \tag{A.4.22}$$

$$\begin{gathered}
\mathcal{S}_\ell^{-} := \mathcal{S}_{\ell,1}^{-} \cup \mathcal{S}_{\ell,2}^{-}, \quad \text{with} \quad \mathcal{S}_{\ell,1}^{-} := \mathbb{N}\times\mathbb{R}\times\mathbb{R} \quad \text{and} \quad \mathcal{S}_{\ell,2}^{-} := \{0\}\times(-\infty,\ell]\times(-\infty,\delta], \\
\mathcal{S}_\ell^{\mathcal{J},-} := \{0\}\times(-\infty,\ell)\times(-\infty,\delta], \qquad \mathcal{S}_\ell^{0,-} := \mathcal{S}_\ell^{-}\setminus\mathcal{S}_\ell^{\mathcal{J},-} = \mathcal{S}_{\ell,1}^{0,-} \cup \mathcal{S}_{\ell,2}^{0,-}, \\
\text{with} \quad \mathcal{S}_{\ell,1}^{0,-} = \mathcal{S}_{\ell,1}^{-}\setminus\mathcal{S}_\ell^{\mathcal{J},-} \quad \text{and} \quad \mathcal{S}_{\ell,2}^{0,-} = \mathcal{S}_{\ell,2}^{-}\setminus\mathcal{S}_\ell^{\mathcal{J},-},
\end{gathered} \tag{A.4.23}$$

and see that both $\mathcal{S}_\ell^{+}$ and $\mathcal{S}_\ell^{-}$ form a closed set in $\mathcal{D}$.[14] Consequently, for each of these domains, the first entry time defined by

$$\tau_{\mathcal{S}_\ell^{\pm}} := \inf\big\{t \geq 0 : \; Z_t \in \mathcal{S}_\ell^{\pm}\big\} \tag{A.4.24}$$

is a stopping time. Furthermore, the finiteness of the first moment of the exponential distribution for any intensity parameter $\vartheta > 0$ implies the $\mathbb{P}_z^Z$-almost sure finiteness of $\tau_{\mathcal{S}_\ell^{\pm}}$ for any $z = (n,x,\delta)$, where $\mathbb{P}_z^Z$ refers to the measure having initial distribution $Z_0 = z$. Using this notation, we can therefore follow the line of the arguments developed in the proof of Proposition 4.2 and re-express the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$ as solutions of appropriate stopping problems:

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\vartheta,x;\ell) = \widehat{V}_0^{\pm}\big((0,x,0)\big) \qquad \text{and} \qquad \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\vartheta,x;\ell) = \widehat{V}_{\mathcal{J}}^{\pm}\big((0,x,0)\big), \tag{A.4.25}$$

where the value functions $\widehat{V}_0^{\pm}(\cdot)$ and $\widehat{V}_{\mathcal{J}}^{\pm}(\cdot)$ have the following probabilistic representations:

$$\widehat{V}_0^{\pm}(z) = \mathbb{E}_z^{\mathbb{P}^Z}\left[\mathbb{1}_{\mathcal{S}_{\ell,2}^{0,\pm}}\Big(Z_{\tau_{\mathcal{S}_\ell^{\pm}}}\Big)\right] \qquad \text{and} \qquad \widehat{V}_{\mathcal{J}}^{\pm}(z) = \mathbb{E}_z^{\mathbb{P}^Z}\left[\mathbb{1}_{\mathcal{S}_\ell^{\mathcal{J},\pm}}\Big(Z_{\tau_{\mathcal{S}_\ell^{\pm}}}\Big)\right]. \tag{A.4.26}$$

[14] We emphasize that several choices of a product-metric on $\mathcal{D}$ give the closedness of the set $\mathcal{S}_\ell^{+}$ and $\mathcal{S}_\ell^{-}$. In particular, one may choose on $\mathbb{N}_0$ the following metric

$$d_{\mathbb{N}_0}(m,n) := \begin{cases} 1 + |2^{-m} - 2^{-n}|, & m \neq n, \\ 0, & m = n, \end{cases}$$

and consider the product-metric on $\mathcal{D}$ obtained by combining $d_{\mathbb{N}_0}(\cdot,\cdot)$ on $\mathbb{N}_0$ with the Euclidean metric on $\mathbb{R}$.

Additionally, standard arguments based on the strong Markov property (cf. [PS06], [Ma19], [Ma20]) imply that, for any $\ell \in \mathbb{R}$, the functions $\widehat{V}_0^{\pm}(\cdot)$ and $\widehat{V}_{\mathcal{J}}^{\pm}(\cdot)$ satisfy the following problems

$$\mathcal{A}_Z \widehat{V}_0^{\pm}\big((n,x,\delta)\big) = 0, \qquad \text{on } \mathcal{D} \setminus \mathcal{S}_\ell^{\pm}, \tag{A.4.27}$$
$$\widehat{V}_0^{\pm}\big((n,x,\delta)\big) = \mathbb{1}_{\mathcal{S}_{\ell,2}^{0,\pm}}\big((n,x,\delta)\big), \qquad \text{on } \mathcal{S}_\ell^{\pm}, \tag{A.4.28}$$

and

$$\mathcal{A}_Z \widehat{V}_{\mathcal{J}}^{\pm}\big((n,x,\delta)\big) = 0, \qquad \text{on } \mathcal{D} \setminus \mathcal{S}_\ell^{\pm}, \tag{A.4.29}$$
$$\widehat{V}_{\mathcal{J}}^{\pm}\big((n,x,\delta)\big) = \mathbb{1}_{\mathcal{S}_{\ell}^{\mathcal{J},\pm}}\big((n,x,\delta)\big), \qquad \text{on } \mathcal{S}_\ell^{\pm}. \tag{A.4.30}$$

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t\geq 0}$. To complete the proof, it therefore suffices to note that (for any suitable function $V : \mathcal{D} \to \mathbb{R}$) the infinitesimal generator $\mathcal{A}_Z$ can be re-expressed as[15]

$$\begin{aligned} \mathcal{A}_Z V(z) &= \mathcal{A}_N^n V\big((n,x,\delta)\big) + \mathcal{A}_X^x V\big((n,x,\delta)\big) && \text{(A.4.31)}\\ &= \vartheta\,\Big(V\big((n+1,x,\delta)\big) - V\big((n,x,\delta)\big)\Big) + \mathcal{A}_X^x V\big((n,x,\delta)\big), && \text{(A.4.32)} \end{aligned}$$

where $\mathcal{A}_N$ denotes the infinitesimal generator of the Poisson process $(N_t)_{t\geq 0}$ and the notation $\mathcal{A}_N^n$ and $\mathcal{A}_X^x$ is used to indicate that the generators are applied to n and x respectively. Indeed, recovering $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{\pm}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{\pm}}\big)(\cdot)$ via (A.4.25) while noting Relation (A.4.32) and the fact that for any $x \in \mathbb{R}$ and $\delta \in \mathbb{R}$ we have

$$\widehat{V}_0^{\pm}\big((1,x,\delta)\big) = 0 \qquad \text{and} \qquad \widehat{V}_{\mathcal{J}}^{\pm}\big((1,x,\delta)\big) = 0 \tag{A.4.33}$$

finally leads to Problems (4.3.30)-(4.3.32) and (4.3.33)-(4.3.35), respectively. □

4.7.3 Appendix C: Proofs – Hyper-Exponential Jump-Diffusions

Proof of Proposition 4.4. We only provide a proof for the upside first-passage probabilities, i.e. for the functions $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{+}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{+}}\big)(\cdot)$, and note that the downside first-passage probabilities can be derived analogously.

We begin by fixing $\vartheta > 0$, $\ell \in \mathbb{R}$ and showing that the maturity-randomized first-passage probabilities $\mathcal{LC}\big(u_X^{\mathcal{E}_0^{+}}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^{+}}\big)(\cdot)$ have the structure described in (4.3.60) with coefficients $(\underline{v}_{0,k})_{k\in\{1,\ldots,m+1\}}$ and $(\underline{v}_{\mathcal{J},k})_{k\in\{1,\ldots,m+1\}}$ satisfying the linear equations in (4.3.61). Here, we first note that the same arguments as in the proof of Theorem 3.2. in [CK11] imply that the general solution to OIDE (4.3.30) takes the form

$$\tilde{V}(\vartheta,x;\ell) = \sum_{k=1}^{m+1} \underline{w}_k^{\star} e^{\beta_{k,\vartheta}\cdot x} + \sum_{h=1}^{n+1} \overline{w}_h^{\star} e^{\gamma_{h,\vartheta}\cdot x}, \tag{A.4.34}$$

or, equivalently,

$$V(\vartheta,x;\ell) = \sum_{k=1}^{m+1} \underline{w}_k e^{\beta_{k,\vartheta}\cdot(x-\ell)} + \sum_{h=1}^{n+1} \overline{w}_h e^{\gamma_{h,\vartheta}\cdot(x-\ell)}. \tag{A.4.35}$$

Although both expressions (A.4.34) and (A.4.35) lead to equivalent results, we choose to follow Ansatz (A.4.35) since it will allow us to separate the dependency of the first-passage level ℓ from the remaining

[15] Here again, we have implicitly used the quasi-left-continuity of the process $(X_t)_{t\geq 0}$, which follows from the Feller property.

parts. This last property will prove useful in subsequent computations discussed, for instance, in Section 4.5. Next, the boundedness of the functions $\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\cdot)$ and $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^+}\big)(\cdot)$ gives that for both of these functions one must have

$$\overline{w}_1 = \ldots = \overline{w}_{n+1} = 0. \tag{A.4.36}$$

Therefore, the functional form (4.3.60) is obtained. We now determine the coefficients $(\underline{v}_{0,k})_{k\in\{1,\ldots,m+1\}}$ and $(\underline{v}_{\mathcal{J},k})_{k\in\{1,\ldots,m+1\}}$. First, the continuous-fit conditions[16]

$$\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,\ell-;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,\ell;\ell) \quad \text{and} \quad \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^+}\big)(\vartheta,\ell-;\ell) = \mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^+}\big)(\vartheta,\ell;\ell) \tag{A.4.37}$$

imply that

$$\sum_{k=1}^{m+1} \underline{v}_{0,k} = 1 \quad \text{and} \quad \sum_{k=1}^{m+1} \underline{v}_{\mathcal{J},k} = 0 \tag{A.4.38}$$

must hold, respectively. Therefore, to fully determine the coefficients $(\underline{v}_{0,k})_{k\in\{1,\ldots,m+1\}}$ and $(\underline{v}_{\mathcal{J},k})_{k\in\{1,\ldots,m+1\}}$ at least m additional equations are required in each case. We derive these equations by substituting (4.3.60) back in OIDE (4.3.30). Here, we focus on $\mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\cdot)$ and will briefly comment on $\mathcal{LC}\big(u_X^{\mathcal{E}_{\mathcal{J}}^+}\big)(\cdot)$ afterwards. To start, we derive for $x < \ell$ that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x+y;\ell) f_{J_1}(y)\,dy \\
&= \int_{-\infty}^{0} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x+y;\ell)\left(\sum_{j=1}^{n} q_j\eta_j e^{\eta_j y}\right) dy + \int_{0}^{\ell-x} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x+y;\ell)\left(\sum_{i=1}^{m} p_i\xi_i e^{-\xi_i y}\right) dy \\
&\qquad + \int_{\ell-x}^{\infty} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x+y;\ell)\left(\sum_{i=1}^{m} p_i\xi_i e^{-\xi_i y}\right) dy \\
&= \sum_{j=1}^{n} q_j\eta_j e^{-\eta_j x}\int_{-\infty}^{x} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,z;\ell) e^{\eta_j z}dz + \sum_{i=1}^{m} p_i\xi_i e^{\xi_i x}\int_{x}^{\ell} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,z;\ell) e^{-\xi_i z}dz \\
&\qquad + \sum_{i=1}^{m} p_i\xi_i e^{\xi_i x}\int_{\ell}^{\infty} \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,z;\ell) e^{-\xi_i z}dz \\
&= \sum_{k=1}^{m+1}\sum_{j=1}^{n} q_j\eta_j e^{-\eta_j x}\underline{v}_{0,k}\int_{-\infty}^{x} e^{(\eta_j+\beta_{k,\vartheta})z}\, e^{-\beta_{k,\vartheta}\cdot\ell}dz + \sum_{k=1}^{m+1}\sum_{i=1}^{m} p_i\xi_i e^{\xi_i x}\underline{v}_{0,k}\int_{x}^{\ell} e^{-(\xi_i-\beta_{k,\vartheta})z}\, e^{-\beta_{k,\vartheta}\cdot\ell}dz
\end{aligned} \tag{A.4.39}$$

$$= \sum_{k=1}^{m+1}\sum_{j=1}^{n}\frac{q_j\eta_j}{\eta_j+\beta_{k,\vartheta}}\underline{v}_{0,k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)} + \sum_{k=1}^{m+1}\sum_{i=1}^{m}\frac{p_i\xi_i}{\xi_i-\beta_{k,\vartheta}}\underline{v}_{0,k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)} - \sum_{k=1}^{m+1}\sum_{i=1}^{m}\frac{p_i\xi_i}{\xi_i-\beta_{k,\vartheta}}\underline{v}_{0,k}\, e^{-\xi_i(\ell-x)}. \tag{A.4.40}$$

[16] Note that these continuous-fit conditions are guaranteed by $\sigma_X > 0$ and the finite jump activity characterizing compound Poisson processes, hence hyper-exponential jump-diffusions. More details can be found, for instance, in [Vo05].

Therefore, using the relations

$$\partial_x \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x;\ell) = \sum_{k=1}^{m+1} \underline{v}_{0,k}\, \beta_{k,\vartheta}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)} \quad \text{and} \quad \partial_x^2 \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x;\ell) = \sum_{k=1}^{m+1} \underline{v}_{0,k} (\beta_{k,\vartheta})^2 e^{\beta_{k,\vartheta}\cdot(x-\ell)} \tag{A.4.41}$$

we arrive at the following equation for $x < \ell$:

$$\begin{aligned}
0 &= \mathcal{A}_X \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x;\ell) - \vartheta\, \mathcal{LC}\big(u_X^{\mathcal{E}_0^+}\big)(\vartheta,x;\ell) \\
&= \sum_{k=1}^{m+1} \underline{v}_{0,k} e^{\beta_{k,\vartheta}\cdot(x-\ell)} \big(\Phi_X(\beta_{k,\vartheta}) - \vartheta\big) - \lambda \sum_{i=1}^{m} p_i e^{-\xi_i(\ell-x)} \left(\sum_{k=1}^{m+1} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}} \underline{v}_{0,k} \right) \\
&= -\lambda \sum_{i=1}^{m} p_i e^{-\xi_i(\ell-x)} \left(\sum_{k=1}^{m+1} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}} \underline{v}_{0,k} \right).
\end{aligned} \tag{A.4.42}$$

Since $\xi_1, \ldots, \xi_m$ are different from each other, we conclude that the following equation must hold:

$$\sum_{k=1}^{m+1} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}} \underline{v}_{0,k} = 0, \qquad \forall i \in \{1, \ldots, m\}. \tag{A.4.43}$$

Hence, combining (A.4.38) and (A.4.43) results in $\underline{\mathbf{A}}_\vartheta \underline{\mathbf{v}}_0 = (1, \mathbf{0}_m)^\intercal$ immediately.

To derive the corresponding system of equations for $\mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^+}\big)(\cdot)$, we proceed as above. First, reproducing the derivation of the integral term, provides the following result for $x < \ell$:

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{LC}\big(u_X^{\mathcal{E}_\mathcal{J}^+}\big)(\vartheta, x+y;\ell) f_{J_1}(y) dy = \sum_{k=1}^{m+1} \sum_{j=1}^{n} \frac{q_j \eta_j}{\eta_j + \beta_{k,\vartheta}}\, \underline{v}_{\mathcal{J},k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)} + \sum_{k=1}^{m+1} \sum_{i=1}^{m} \frac{p_i \xi_i}{\xi_i - \beta_{k,\vartheta}}\, \underline{v}_{\mathcal{J},k}\, e^{\beta_{k,\vartheta}\cdot(x-\ell)} \\
- \sum_{k=1}^{m+1} \sum_{i=1}^{m} \frac{p_i \xi_i}{\xi_i - \beta_{k,\vartheta}}\, \underline{v}_{\mathcal{J},k}\, e^{-\xi_i(\ell-x)} + \sum_{i=1}^{m} p_i e^{-\xi_i(\ell-x)}.
\end{aligned} \tag{A.4.44}$$

Then, inserting the latter expression in OIDE (4.3.30) gives that

$$0 = -\lambda \sum_{i=1}^{m} p_i e^{-\xi_i(\ell-x)} \left(\sum_{k=1}^{m+1} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}} \underline{v}_{\mathcal{J},k} - 1 \right). \tag{A.4.45}$$

This finally implies that

$$\sum_{k=1}^{m+1} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}} \underline{v}_{\mathcal{J},k} = 1, \qquad \forall i \in \{1, \ldots, m\}, \tag{A.4.46}$$

which immediately results, together with (A.4.38), in $\underline{\mathbf{A}}_\vartheta \underline{\mathbf{v}}_\mathcal{J} = (0, \mathbf{1}_m)^\intercal$.

To conclude, we note that the uniqueness of all the vectors $\underline{\mathbf{v}}_0$, $\underline{\mathbf{v}}_\mathcal{J}$, $\overline{\mathbf{v}}_0$ and $\overline{\mathbf{v}}_\mathcal{J}$ follows from the invertibility of the matrices $\underline{\mathbf{A}}_\vartheta$ and $\overline{\mathbf{A}}_\vartheta$ (cf. [CK11]). $\square$

Proof of Proposition 4.5. Our proof mainly relies on arguments introduced in [CCW09] (cf. also [CYY13]) and focuses, as in the proof of Propositions 4.4, on the upside first-passage probabilities. However, we note that the same techniques can be applied to derive the corresponding downside results.

We start by considering, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$, and $\vartheta > 0$ the function

$$b \mapsto \mathbb{E}_x^{\mathbb{P}^X} \left[\exp\left\{ -\vartheta \tau_\ell^{X,+} + b X_{\tau_\ell^{X,+}} \right\} \right] \tag{A.4.47}$$

on the domain $\mathcal{D} := \{ b \in \mathbb{C} : \mathrm{Re}(b) \geq 0 \}$ and recall that the overshoot distribution is conditionally memoryless and independent of the first-passage time provided the overshoot is greater than zero and the exponential type of the jump distribution is specified, i.e. we have, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$ and any $t > 0$, $y > 0$,

$$\mathbb{P}_x^X \left(X_{\tau_\ell^{X,+}} - \ell \geq y \,\middle|\, X_{\tau_\ell^{X,+}} > \ell,\, J_{N_{\tau_\ell^{X,+}}} \sim \mathrm{Exp}(\xi_i) \right) = e^{-\xi_i y}, \tag{A.4.48}$$

and

$$\begin{aligned} &\mathbb{P}_x^X \left(\tau_\ell^{X,+} \leq t,\, X_{\tau_\ell^{X,+}} - \ell \geq y \,\middle|\, X_{\tau_\ell^{X,+}} > \ell,\, J_{N_{\tau_\ell^{X,+}}} \sim \mathrm{Exp}(\xi_i) \right) \\ &\quad = \mathbb{P}_x^X \left(\tau_\ell^{X,+} \leq t \,\middle|\, X_{\tau_\ell^{X,+}} > \ell,\, J_{N_{\tau_\ell^{X,+}}} \sim \mathrm{Exp}(\xi_i) \right) \cdot \mathbb{P}_x^X \left(X_{\tau_\ell^{X,+}} - \ell \geq y \,\middle|\, X_{\tau_\ell^{X,+}} > \ell,\, J_{N_{\tau_\ell^{X,+}}} \sim \mathrm{Exp}(\xi_i) \right). \end{aligned} \tag{A.4.49}$$

This was already derived in [Ca09] and implies, in particular, that for any purely imaginary number $b \in \mathbb{C}$

$$\begin{aligned} \mathbb{E}_x^{\mathbb{P}^X} \left[\exp\left\{ -\vartheta \tau_\ell^{X,+} + b X_{\tau_\ell^{X,+}} \right\} \right] &= e^{b \cdot \ell}\, \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_0^+} \right] + e^{b \cdot \ell} \sum_{i=1}^m \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_i^+} \right] \int_0^\infty \xi_i e^{-(\xi_i - b) y}\, dy, \\ &= e^{b \cdot \ell}\, \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_0^+} \right] + e^{b \cdot \ell} \sum_{i=1}^m \frac{\xi_i}{\xi_i - b}\, \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_i^+} \right]. \end{aligned} \tag{A.4.50}$$

Combining this identity with an analytic continuation argument will allow us to derive the required system of equations. Indeed, we first note that, for any $x \in \mathbb{R}$, $\vartheta > 0$ and any purely imaginary number $b \in \mathbb{C}$, the process $(M_t)_{t \geq 0}$ defined via

$$M_t := e^{-\vartheta t + b X_t} - e^{bx} - \big(\Phi_X(b) - \vartheta \big) \int_0^t e^{-\vartheta s + b X_s} ds \tag{A.4.51}$$

is a zero-mean $\mathbb{P}_x^X$-martingale and obtain, for $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$, via the optional sampling theorem that

$$\begin{aligned} 0 &= \mathbb{E}_x^{\mathbb{P}^X} \left[\exp\left\{ -\vartheta \tau_\ell^{X,+} + b X_{\tau_\ell^{X,+}} \right\} \right] - e^{bx} - \big(\Phi_X(b) - \vartheta \big)\, \mathbb{E}_x^{\mathbb{P}^X} \left[\int_0^{\tau_\ell^{X,+}} e^{-\vartheta s + b X_s} ds \right] \\ &= \underbrace{e^{b \cdot \ell}\, \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_0^+} \right] + e^{b \cdot \ell} \sum_{i=1}^m \frac{\xi_i}{\xi_i - b}\, \mathbb{E}_x^{\mathbb{P}^X} \left[e^{-\vartheta \tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_i^+} \right] - e^{bx} - \big(\Phi_X(b) - \vartheta \big)\, \mathbb{E}_x^{\mathbb{P}^X} \left[\int_0^{\tau_\ell^{X,+}} e^{-\vartheta s + b X_s} ds \right]}_{=: g(b)}. \end{aligned} \tag{A.4.52}$$

Hence, if one defines a new function by $G(b) := \prod_{i=1}^{m} (\xi_i - b) \cdot g(b)$, one easily sees that $G(\cdot)$ is well-defined and (as a function of b) analytic on the full domain $\mathcal{D}$. Therefore, by the identity theorem for analytic functions (cf. [Ru87]), we must have that $G(b) \equiv 0$ for all $b \in \mathcal{D}$. Accordingly, we must have that $g(b) \equiv 0$ for all $b \in \mathcal{D} \setminus \{\xi_1, \ldots, \xi_m\}$. This finally allows us to replace b in Equation (A.4.52) by the positive roots $\beta_{1,\vartheta}, \ldots, \beta_{m+1,\vartheta}$ to obtain that

$$e^{\beta_{k,\vartheta}\cdot(x-\ell)} = \mathbb{E}_x^{\mathbb{P}^X}\left[e^{-\vartheta\tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_0^+}\right] + \sum_{i=1}^{m} \frac{\xi_i}{\xi_i - \beta_{k,\vartheta}}\, \mathbb{E}_x^{\mathbb{P}^X}\left[e^{-\vartheta\tau_\ell^{X,+}} \mathbb{1}_{\mathcal{E}_i^+}\right], \qquad \forall k \in \{1, \ldots, m+1\}, \tag{A.4.53}$$

which gives, in view of (4.3.69), the required system of equations. □

Proof of Proposition 4.6. Our derivations are based on the results obtained in Theorem 2.1 and Theorem 2.2 in [CYY13]. Indeed, combining first the results of Theorem 2.2 with (4.3.69) and Proposition 4.5 gives that for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^+}$ and $\vartheta > 0$ we have

$$\mathcal{LC}(u_X^{\mathcal{E}_0^+})(\vartheta, x; \ell) = \mathbf{G}^+ \, e^{\beta_{1,\vartheta}\cdot(x-\ell)} - \mathbf{G}^+ \cdot \sum_{k=2}^{m+1}\left(\sum_{s=1}^{m} \frac{d_{k,s}^+}{\xi_s - \beta_{1,\vartheta}}\right) \cdot e^{\beta_{k,\vartheta}\cdot(x-\ell)}, \tag{A.4.54}$$

and, for $i \in \{1, \ldots, m\}$,

$$\mathcal{LC}(u_X^{\mathcal{E}_i^+})(\vartheta, x; \ell) = -\frac{\mathbf{G}^+}{\xi_i} \frac{\mathbf{C}_\vartheta^+(\xi_i)}{\left(\mathbf{B}^+\right)'(\xi_i)}\, e^{\beta_{1,\vartheta}\cdot(x-\ell)} + \frac{1}{\xi_i}\sum_{k=2}^{m+1}\left(d_{k,i}^+ + \mathbf{G}^+ \, \frac{\mathbf{C}_\vartheta^+(\xi_i)}{\left(\mathbf{B}^+\right)'(\xi_i)} \sum_{s=1}^{m} \frac{d_{k,s}^+}{\xi_s - \beta_{1,\vartheta}}\right) \cdot e^{\beta_{k,\vartheta}\cdot(x-\ell)}, \tag{A.4.55}$$

with $\mathbf{G}^+ := \frac{\mathbf{B}^+(\beta_{1,\vartheta})}{\mathbf{C}_\vartheta^+(\beta_{1,\vartheta})}$ and $\mathbf{B}^+(\cdot)$, $\mathbf{C}_\vartheta^+(\cdot)$ and $d_{i,j}^+$ defined as in (4.3.76) and (4.3.77). Therefore, comparing Result (A.4.54) with the corresponding results in Proposition 4.4 already gives that

$$\underline{v}_{0,1} = \mathbf{G}^+ \qquad \text{and} \qquad \underline{v}_{0,k} = -\mathbf{G}^+ \cdot \sum_{s=1}^{m} \frac{d_{k,s}^+}{\xi_s - \beta_{1,\vartheta}}, \quad k \in \{2, \ldots, m+1\}, \tag{A.4.56}$$

and substituting this identity back in (A.4.55) finally gives the results (4.3.74) and (4.3.75).

Deriving the results for the downside case can be done in the same way. First, combining Theorem 2.1 in [CYY13] with (4.3.69) and Proposition 4.5 gives, for $\ell \in \mathbb{R}$, $x \in \mathbb{R} \setminus \overline{\mathcal{H}_\ell^-}$ and $\vartheta > 0$, that

$$\mathcal{LC}(u_X^{\mathcal{E}_0^-})(\vartheta, x; \ell) = \mathbf{G}^- \, e^{\gamma_{1,\vartheta}\cdot(x-\ell)} - \mathbf{G}^- \cdot \sum_{k=2}^{n+1}\left(\sum_{s=1}^{n} \frac{d_{k,s}^-}{\eta_s + \gamma_{1,\vartheta}}\right) \cdot e^{\gamma_{k,\vartheta}\cdot(x-\ell)} \tag{A.4.57}$$

and, for $j \in \{1, \ldots, n\}$,

$$\begin{aligned}\mathcal{LC}(u_X^{\mathcal{E}_j^-})(\vartheta, x; \ell) = {} & (-1)^n \, \frac{\mathbf{G}^-}{\eta_j} \frac{\mathbf{C}_\vartheta^-(\eta_j)}{\left(\mathbf{B}^-\right)'(-\eta_j)}\, e^{\gamma_{1,\vartheta}\cdot(x-\ell)} \\ & + \frac{1}{\eta_j}\sum_{k=2}^{n+1}\left(d_{k,j}^- + (-1)^{n-1}\,\mathbf{G}^- \, \frac{\mathbf{C}_\vartheta^-(\eta_j)}{\left(\mathbf{B}^-\right)'(-\eta_j)} \sum_{s=1}^{n} \frac{d_{k,s}^-}{\eta_s + \gamma_{1,\vartheta}}\right) \cdot e^{\gamma_{k,\vartheta}\cdot(x-\ell)}\end{aligned} \tag{A.4.58}$$

with $\mathbf{G}^{-} := (-1)^{n} \frac{\mathbf{B}^{-}(\gamma_{1,\vartheta})}{\mathbf{C}_{\vartheta}^{-}(-\gamma_{1,\vartheta})}$ and $\mathbf{B}^{-}(\cdot)$, $\mathbf{C}_{\vartheta}^{-}(\cdot)$ and $d_{i,j}^{-}$ defined as in (4.3.80) and (4.3.81). Therefore, comparing Result (A.4.54) with the corresponding results in Proposition 4.4 already gives that

$$\overline{v}_{0,1} = \mathbf{G}^{-} \qquad \text{and} \qquad \overline{v}_{0,k} = -\mathbf{G}^{-} \cdot \sum_{s=1}^{n} \frac{d_{k,s}^{-}}{\eta_s + \gamma_{1,\vartheta}}, \quad k \in \{2, \ldots, n+1\} \tag{A.4.59}$$

holds and substituting this identity back in (A.4.58) finally gives (4.3.78) and (4.3.79). □

Acknowledgements: The authors would like to thank Sergei Levendorskii, Tadeusz Czernik, Max Nendel, Carlo Sala, Giovanni Barone-Adesi, Matteo Burzoni, Johannes Wiesel, Felix-Benedikt Liebrich and the participants of the 9th General AMaMeF Conference, the 2019 Vienna Congress on Mathematical Finance (VCMF 2019), and the 2019 Quantitative Methods in Finance Conference (QMF 2019) for their valuable comments and suggestions.

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Chapter 5

Geometric Step Options with Jumps: Parity Relations, PIDEs, and Semi-Analytical Pricing

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Abstract

The present article studies geometric step options in exponential Lévy markets. Our contribution is manifold and extends several aspects of the geometric step option pricing literature. First, we provide symmetry and parity relations and derive various characterizations for both European-type and American-type geometric double barrier step options. In particular, we are able to obtain a jump-diffusion disentanglement for the early exercise premium of American-type geometric double barrier step contracts and its maturity-randomized equivalent as well as to characterize the diffusion and jump contributions to these early exercise premiums separately by means of partial integro-differential equations and ordinary integro-differential equations. As an application of our characterizations, we derive semi-analytical pricing results for (regular) European-type and American-type geometric down-and-out step call options under hyper-exponential jump-diffusion models. Lastly, we use the latter results to discuss the early exercise structure of geometric step options once jumps are added and to subsequently provide an analysis of the impact of jumps on the price and hedging parameters of (European-type and American-type) geometric step contracts.

Keywords: Geometric Step Options, American-Type Options, Lévy Markets, Jump-Diffusion Disentanglement, Maturity-Randomization.

MSC (2010) Classification: 91-08, 91B25, 91B70, 91G20, 91G60, 91G80.

JEL Classification: C32, C61, C63, G13.

5.1 Introduction

Since their introduction in the seminal article of Linetsky (cf. [Li99]) and their generalization in the subsequent work of Davydov and Linetsky (cf. [DL02]) geometric step options have constantly gained attention in both the financial industry and the academic literature (cf. [CCW10], [CMW13], [XY13], [WZ16], [WZB17], [DLM20]). As a whole class of financial contracts written on an underlying asset, these options have the particularity to cumulatively and proportionally loose or gain value when the underlying asset price stays below or above a predetermined threshold and consequently offer a continuum of alternatives between standard options and (standard) barrier options. Especially when compared with the latter options, geometric step contracts bring clear advantages: Due to their immediate cancellation (or activation) when the barrier level is breached, (standard) barrier options are extremely sensitive to any (temporary) change in the underlying asset price near the barrier so that (delta-)hedging is not reasonably feasible in this region. Additionally, the immediate knock-out (or knock-in) feature inherent to (standard) barrier options may incentivize influential market participants to manipulate the underlying asset price close to the barrier, hence triggering cancellation (or activation) of these options. Switching from an immediate to a cumulative and proportional knock-out (or knock-in) feature instead substantially helps addressing these concerns. Indeed, in contrast to (standard) barrier options, the delta of geometric step contracts does not explode and is even continuous at the barrier. This already allows for typical delta-hedges across the barrier level. Furthermore, since it is more difficult to control underlying asset prices over an extended period of time, geometric step options are more robust to temporary market manipulations and therefore better protect their holders against adverse actions of market participants in the underlying asset.

The present article studies (European-type and American-type) geometric step contracts under exponential Lévy dynamics. Our paper's contribution is manifold and extends several aspects of the geometric step option pricing literature: Firstly, we establish symmetry and parity relations for geometric double barrier step contracts under exponential Lévy models. Since standard options are naturally embedded in the whole class of geometric double barrier step options, these results generalize in particular the ones obtained in [FM06], [FM14]. Secondly, we derive various characterizations for European-type and American-type geometric double barrier step contracts as well as for their respective maturity-randomized quantities. Most notably, we are able to derive a jump-diffusion disentanglement for the early exercise premium of American-type geometric double barrier step options and its maturity-randomized equivalent as well as to characterize the diffusion and jump contributions to these early exercise premiums separately by means of partial integro-differential equations (PIDEs) and ordinary integro-differential equations (OIDEs). Our results translate the formalism introduced in the third research article (cf. [FMV19]) to the setting of geometric double barrier step contracts and generalize at the same time the ideas introduced in [CYY13], [LV17] and [CV18] to Lévy-driven markets. Next, as an application of these characterizations, we derive semi-analytical pricing results for (regular) European-type and American-type geometric down-and-out step call options under hyper-exponential jump-diffusion processes.[1] Although semi-analytical pricing results for European-type geometric step options were already obtained by other authors under similar asset dynamics (cf. [CCW10], [WZ16], [WZB17]), we note that these results employed double Laplace transform techniques while our method only relies on a one-dimensional Laplace(-Carson) transform. Additionally, the current geometric step option pricing literature seems to either study the Black & Scholes framework (cf. [BS73]) or only European-type geometric step options under more advanced models. To the best of our knowledge, we

[1] It is worth recalling that hyper-exponential jump-diffusion processes are particularly suitable for financial modeling since they are able to provide arbitrarily close approximations to Lévy processes having a completely monotone jump density. The latter processes form an important class of Lévy models and include popular market dynamics such as Variance Gamma (VG) processes (cf. [MS90], [MCC98]), the CGMY model (cf. [CGMY02]), and Normal Inverse Gaussian (NIG) processes (cf. [BN97]).

are therefore the first to provide characterizations as well as (tractable) pricing results for American-type geometric step options. Lastly, we discuss the early exercise structure of geometric step options once jumps are added and subsequently provide an analysis of the impact of jumps on the price and hedging parameters of (European-type and American-type) geometric step contracts. As of now, no clear investigation of this sensitivity to jumps has been provided in the geometric step option pricing literature, which is mainly due to the scarcity of publications dealing with (American-type) geometric step options with jumps.

The remaining of this paper is structured as follows: In Section 5.2, we introduce (European-type and American-type) geometric step options under exponential Lévy markets and discuss symmetry and parity relations as well as PIDE and OIDE characterizations of these options. Section 5.3 deals with geometric step contracts under hyper-exponential jump-diffusion models. Here, semi-analytical pricing results for both European-type and American-type contracts are derived by combining the derivations of Section 5.2 with certain properties of the hyper-exponential distribution. These theoretical results are subsequently exemplified in Section 5.4, where structural and numerical properties of (regular) geometric down-and-out step call options with jumps are illustrated and a comparison to the respective results in the standard Black & Scholes framework is provided. The paper concludes with Section 5.5. All proofs and complementary results are presented in the appendices (Appendix A and B; Section 5.6).

5.2 Geometric Step Options and Exponential Lévy Markets

5.2.1 General Framework

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ – a chosen risk-neutral probability space[2] –, whose filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and consider two assets, a deterministic savings account $(B_t(r))_{t\geq 0}$ satisfying

$$B_t(r) = e^{rt}, \qquad r \geq 0,\, t \geq 0, \tag{5.2.1}$$

and a risky asset $(S_t)_{t\geq 0}$, whose price dynamics, under $\mathbb{Q}$, are described by the following (ordinary) exponential Lévy model

$$S_t = S_0 e^{X_t}, \qquad S_0 > 0,\, t \geq 0. \tag{5.2.2}$$

Here, the process $(X_t)_{t\geq 0}$ is an $\mathbf{F}$-Lévy process associated with a triplet (b_X, σ_X^2, Π_X), i.e. a càdlàg (right-continuous with left limits) process having independent and stationary increments and Lévy-exponent $\Psi_X(\cdot)$ defined, for $\theta \in \mathbb{R}$, by

$$\Psi_X(\theta) := -\log\left(\mathbb{E}^{\mathbb{Q}}\left[e^{i\theta X_1}\right]\right) = -ib_X\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\leq 1\}}\right)\Pi_X(dy), \tag{5.2.3}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ refers to expectation with respect to the measure $\mathbb{Q}$. Numerous models in the financial literature fall into this framework. Important examples include hyper-exponential jump-diffusion (HEJD) models (cf. [Ko02], [Ca09]), Variance Gamma (VG) processes (cf. [MS90], [MCC98]), the CGMY model (cf. [CGMY02]) as well as Generalized Hyperbolic (GH) processes such as the popular Normal Inverse Gaussian (NIG) model (cf. [BN97]).

Applying standard results (cf. [Sa99], [Ap09]), allows us to decompose $(X_t)_{t\geq 0}$ in terms of its diffusion and

[2] It is well-known that exponential Lévy markets are incomplete as defined by Harrison and Pliska (cf. [HP81]). Specifying or discussing a particular choice of risk-neutral measure is not the sake of this article. Instead, we assume that a pricing measure under which our model has the required dynamics was previously fixed.

jump parts as

$$X_t = b_X t + \sigma_X W_t + \int_{\mathbb{R}} y \; \bar{N}_X(t, dy), \qquad t \geq 0, \tag{5.2.4}$$

where $(W_t)_{t\geq 0}$ denotes an **F**-Brownian motion, and N_X refers to an independent Poisson random measure on $[0,\infty) \times \mathbb{R} \setminus \{0\}$ that has intensity measure given by Π_X. Here, we use for $t \geq 0$ and any Borel set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ the following notation:

$$\begin{aligned} N_X(t, A) &:= N_X((0,t] \times A), \\ \tilde{N}_X(dt, dy) &:= N_X(dt, dy) - \Pi_X(dy)dt, \\ \bar{N}_X(dt, dy) &:= \begin{cases} \tilde{N}_X(dt, dy), & \text{if } |y| \leq 1, \\ N_X(dt, dy), & \text{if } |y| > 1. \end{cases} \end{aligned}$$

Additionally, the Laplace exponent of the Lévy process $(X_t)_{t\geq 0}$ can be defined for any $\theta \in \mathbb{R}$ satisfying $\mathbb{E}^{\mathbb{Q}}\left[e^{\theta X_1}\right] < \infty$ and is then recovered from $\Psi_X(\cdot)$ via the following identity:

$$\Phi_X(\theta) := -\Psi_X(-i\theta) = b_X\theta + \frac{1}{2}\sigma_X^2\theta^2 - \int_{\mathbb{R}} \left(1 - e^{\theta y} + \theta y \mathbb{1}_{\{|y|\leq 1\}}\right)\Pi_X(dy). \tag{5.2.5}$$

In the sequel, we always assume that $\Phi_X(\cdot)$ is at least for $\theta = 1$ well-defined or, equivalently, that the price process $(S_t)_{t\geq 0}$ is integrable. Additionally, we assume that the asset $(S_t)_{t\geq 0}$ pays a proportional dividend with constant rate $\delta \geq 0$. In terms of the asset dynamics, this implies that the discounted cum-dividend price process $(e^{-(r-\delta)t}S_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}$, which then requires that

$$\Phi_X(1) = r - \delta. \tag{5.2.6}$$

In particular, rewriting (5.2.6) allows us to recover the following expression for b_X:

$$b_X = r - \delta - \frac{1}{2}\sigma_X^2 + \int_{\mathbb{R}} \left(1 - e^y + y\mathbb{1}_{\{|y|\leq 1\}}\right)\Pi_X(dy). \tag{5.2.7}$$

Such dynamics are typically found when studying foreign exchange markets. In this case, holdings in the foreign currency can earn the foreign risk-free interest rate, which therefore corresponds, for each investment in the foreign currency, to a dividend payment of a certain amount $\delta \geq 0$ (cf. [JC04], [GK83]).

Finally, it should be noted that $(S_t)_{t\geq 0}$ has a Markovian structure. Following standard theory for Markov processes, we therefore recall that its infinitesimal generator is a partial integro-differential operator given, for sufficiently smooth $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, by

$$\begin{aligned} \mathcal{A}_S V(\mathcal{T}, x) &:= \lim_{t \downarrow 0} \frac{\mathbb{E}_x^{\mathbb{Q}}\left[V(\mathcal{T}, S_t)\right] - V(\mathcal{T}, x)}{t} \\ &= \frac{1}{2}\sigma_X^2 x^2 \partial_x^2 V(\mathcal{T}, x) + \Phi_X(1) x \partial_x V(\mathcal{T}, x) \\ &\qquad + \int_{\mathbb{R}} \left[V(\mathcal{T}, xe^y) - V(\mathcal{T}, x) - x(e^y - 1)\partial_x V(\mathcal{T}, x)\right]\Pi_X(dy), \end{aligned} \tag{5.2.8}$$

where $\mathbb{E}_x^{\mathbb{Q}}[\cdot]$ denotes expectation under $\mathbb{Q}_x$, the pricing measure having initial distribution $S_0 = x$. We will extensively make use of these notations in the upcoming sections.

5.2.2 Characterizing Geometric Step Options

As mentioned in the introduction, geometric step options are financial contracts that are written on an underlying asset and that cumulatively and proportionally loose or gain value when the underlying's price stays above or below a certain, predetermined threshold. As such, these contracts are closely linked to the time the asset's price spends above or below a barrier level, so-called occupation times. To fix the notation, we define, for a time $t \geq 0$, the occupation time of asset $(S_t)_{t\geq 0}$ below $(-)$ and above $(+)$ a constant barrier level $\ell > 0$ over the time interval $[0,t]$ via

$$\Gamma^-_{t,\ell} := \int_0^t \mathbb{1}_{(0,\ell)}(S_r)dr, \quad \text{and} \quad \Gamma^+_{t,\ell} := \int_0^t \mathbb{1}_{(\ell,\infty)}(S_r)dr. \tag{5.2.9}$$

In addition, we set, for $\gamma \geq 0$,

$$\Gamma^{\pm}_{t,\ell}(\gamma) := \gamma + \Gamma^{\pm}_{t,\ell} \tag{5.2.10}$$

and allow this way each of the occupation times $\Gamma^-_{t,\ell}$ and $\Gamma^+_{t,\ell}$ to start at a given initial value $\gamma \geq 0$. This generalization proves useful when valuing geometric step options over their entire lifetime. In this case, γ refers to the occupation time the process $(S_t)_{t\geq 0}$ has spent in the respective region from the establishment of the contract until the valuation date under consideration.

As for many other types of options, geometric step options can be found in various styles. Depending on the exercise specification, there exist European-type and American-type geometric step call and put options. Additionally, one can distinguish between "knock-in", "knock-out" as well as "up" and "down" features. Therefore, it is possible to construct a total of 32 different geometric step contracts, all of which can be studied in the unifying framework of geometric double barrier step options. A geometric double barrier step option with initial values $S_0 = x \geq 0$ and $\Gamma^-_{0,L}(\gamma_L) = \gamma_L \geq 0$, $\Gamma^+_{0,H}(\gamma_H) = \gamma_H \geq 0$, strike price $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out/knock-in rates $\rho_L, \rho_H \in \mathbb{R}$ pays off

$$e^{\rho_L \Gamma^-_{t,L}(\gamma_L) + \rho_H \Gamma^+_{t,H}(\gamma_H)}\,(S_t - K)^+ \quad \text{(for a call)} \qquad \text{or} \qquad e^{\rho_L \Gamma^-_{t,L}(\gamma_L) + \rho_H \Gamma^+_{t,H}(\gamma_H)}\,(K - S_t)^+ \quad \text{(for a put)} \tag{5.2.11}$$

at the exercise time $t \geq 0$. Here, any of the barrier levels, $\ell \in \{L, H\}$, is said to be of knock-out type whenever $\rho_\ell \leq 0$, while the case of $\rho_\ell > 0$ is referred to as a knock-in feature.

Using standard valuation principles, probabilistic representations for the value of any type of geometric double barrier step options are readily obtained. For instance, the value of a European-type geometric double barrier knock-out step call defined on the exponential Lévy market (5.2.1), (5.2.2), (5.2.6) and having maturity $\mathcal{T} \geq 0$, initial values $S_0 = x \geq 0$ and $\Gamma^-_{0,L}(\gamma_L) = \gamma_L \geq 0$, $\Gamma^+_{0,H}(\gamma_H) = \gamma_H \geq 0$, strike price $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$ is obtained as

$$\mathcal{DSC}_E\big(\mathcal{T}, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) := \mathbb{E}^{\mathbb{Q}}_x \left[B_{\mathcal{T}}(r)^{-1}\, e^{\rho_L \Gamma^-_{\mathcal{T},L}(\gamma_L) + \rho_H \Gamma^+_{\mathcal{T},H}(\gamma_H)}\, (S_{\mathcal{T}} - K)^+ \right], \tag{5.2.12}$$

where we use the Lévy-exponent $\Psi_X(\cdot)$ to refer to the dynamics of the Lévy process (5.2.4) and therefore to further characterize the dynamics of the underlying price process $(S_t)_{t\geq 0}$ specified in (5.2.2). Similarly, the value of a corresponding American-type geometric double barrier knock-out step call can be shown to have the representation

$$\mathcal{DSC}_A\big(\mathcal{T}, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) := \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}}_x \left[B_\tau(r)^{-1}\, e^{\rho_L \Gamma^-_{\tau,L}(\gamma_L) + \rho_H \Gamma^+_{\tau,H}(\gamma_H)}\, (S_\tau - K)^+ \right], \tag{5.2.13}$$

where $\mathfrak{T}_{[0,\mathcal{T}]}$ denotes the set of stopping times that take values in the interval $[0,\mathcal{T}]$. Here, we note that both values (5.2.12) and (5.2.13) may be understood, for a given pair of times (t,T) satisfying $0 \leq t \leq T < \infty$, as the time-t value of the respective geometric step contract having maturity T, i.e. we usually have in mind that $\mathcal{T} = T - t$ denotes the remaining time to maturity.

At this point, it is important to emphasize that other types of step options exist. Already in his seminal work, Linetsky introduced the class of arithmetic step options as other alternative to barrier options. Compared to standard call and put options, both geometric and arithmetic step options are characterized by an additional adjustment factor. However, while the adjustment factor of geometric step options is given as exponential function of (possibly one of) the occupation times defined in (5.2.10), arithmetic step contracts are characterized by truncated linear adjustments. This implies in particular that, under comparable knock-out rates, arithmetic step contracts will knock-out faster than their geometric counterparts (cf. [Li99], [DL02]). Clearly, our goal is not to discuss results for all existing types of step options. We will therefore mainly focus on geometric double barrier knock-out step calls and leverage on the fact that certain symmetry and parity relations hold between different geometric step contracts. Establishing these relations is the content of the next section.

5.2.3 Symmetry and Parity Relations

To allow for a simultaneous treatment of both European-type and American-type geometric step contracts, we start by introducing, for $T > 0$, any stopping time $\tau \in \mathfrak{T}_{[0,T]}$, initial values $S_0 = x \geq 0$ and $\Gamma^-_{0,L}(\gamma_L) = \gamma_L \geq 0$, $\Gamma^+_{0,H}(\gamma_H) = \gamma_H \geq 0$, strike price $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out/knock-in rates $\rho_L, \rho_H \in \mathbb{R}$, the following quantities:

$$\mathcal{DSC}\big(\tau, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) := \mathbb{E}^{\mathbb{Q}}_x \left[B_\tau(r)^{-1}\, e^{\rho_L \Gamma^-_{\tau,L}(\gamma_L) + \rho_H \Gamma^+_{\tau,H}(\gamma_H)}\, (S_\tau - K)^+ \right], \tag{5.2.14}$$

$$\mathcal{DSP}\big(\tau, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) := \mathbb{E}^{\mathbb{Q}}_x \left[B_\tau(r)^{-1}\, e^{\rho_L \Gamma^-_{\tau,L}(\gamma_L) + \rho_H \Gamma^+_{\tau,H}(\gamma_H)}\, (K - S_\tau)^+ \right]. \tag{5.2.15}$$

Using this notation, the next put-call-duality result can be derived. A proof is provided in Appendix A (cf. Section 5.6.1).

Lemma 5.1 (Duality of Geometric Step Contracts)**.** *Consider an exponential Lévy market, as introduced in (5.2.1), (5.2.2) and (5.2.6), with driving process $(X_t)_{t\geq 0}$ having Lévy exponent given as in (5.2.3). Then, under the notation (5.2.14) and (5.2.15), we have for any $T > 0$ and stopping time $\tau \in \mathfrak{T}_{[0,T]}$ that*

$$\mathcal{DSC}\big(\tau, x, \gamma_L, \gamma_H;\, r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) = \mathcal{DSP}\Big(\tau, K, \gamma_H, \gamma_L; \delta, r, x, \frac{xK}{H}, \frac{xK}{L}, \rho_H, \rho_L, \Psi_Y(\cdot)\Big), \tag{5.2.16}$$

where $\Psi_Y(\cdot)$ represents the Lévy exponent of another Lévy process $(Y_t)_{t\geq 0}$ driving an exponential Lévy market with

$$\Psi_Y(\theta) = \Psi_X(-(\theta + i)) + \Phi_X(1). \tag{5.2.17}$$

In particular, we obtain that the Lévy exponent $\Psi_Y(\cdot)$ is given by

$$\Psi_Y(\theta) = -ib_Y\theta + \frac{1}{2}\sigma_Y^2\theta^2 + \int_{\mathbb{R}} \big(1 - e^{i\theta y} + i\theta y \mathbb{1}_{\{|y|\leq 1\}}\big)\Pi_Y(dy), \tag{5.2.18}$$

where $\left(b_Y, \sigma_Y^2, \Pi_Y\right)$ *are obtained as*

$$b_Y = \delta - r - \frac{1}{2}\sigma_Y^2 + \int_{\mathbb{R}} \left(1 - e^y + y \mathbb{1}_{\{|y| \leq 1\}}\right) \Pi_Y(dy), \tag{5.2.19}$$

$$\sigma_Y^2 = \sigma_X^2, \tag{5.2.20}$$

$$\Pi_Y(dy) = e^{-y}\, \Pi_X(-dy). \tag{5.2.21}$$

Remark 5.1.

Our results in Lemma 5.1 are similar to Lemma 1 in [FM06]. However, while these authors consider standard options, our results hold within the whole class of geometric double barrier step contracts. In particular, since geometric double barrier step options reduce to standard options for $\rho_L = \rho_H = 0$, Lemma 5.1 offers a generalization of the derivations obtained in [FM06]. Additionally, our proof reveals that similar results could be derived for other occupation time derivatives. Due to the focus of our article, we nevertheless refrain from discussing further duality results here.

♦

Combining Lemma 5.1 with few simple transformations allows us to derive duality and symmetry relations for European-type and American-type geometric step options. The results are summarized in the next corollary, whose proof is given in Appendix A (cf. Section 5.6.1).

Corollary 5.1 (Duality and Symmetry of Geometric Step Contracts)**.** *Consider an exponential Lévy market, as introduced in (5.2.1), (5.2.2) and (5.2.6), with driving process* $(X_t)_{t \geq 0}$ *having Lévy exponent given as in (5.2.3). Then, the following duality and symmetry results hold*

$$\mathcal{DSC}_\bullet\big(\mathcal{T}, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) = \mathcal{DSP}_\bullet\Big(\mathcal{T}, K, \gamma_H, \gamma_L; \delta, r, x, \frac{xK}{H}, \frac{xK}{L}, \rho_H, \rho_L, \Psi_Y(\cdot)\Big), \tag{5.2.22}$$

$$\mathcal{DSC}_\bullet\big(\mathcal{T}, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) = xK \cdot \mathcal{DSP}_\bullet\Big(\mathcal{T}, \frac{1}{x}, \gamma_H, \gamma_L; \delta, r, \frac{1}{K}, \frac{1}{H}, \frac{1}{L}, \rho_H, \rho_L, \Psi_Y(\cdot)\Big), \tag{5.2.23}$$

where the Lévy exponents $\Psi_Y(\cdot)$ *is defined as in Lemma 5.1 and* $\bullet$ *refers to the exercise specification of the options, i.e.* $\bullet \in \{E, A\}$.

5.2.4 Geometric Step Options and PIDEs

We next turn to the pricing of geometric double barrier step contracts. As already mentioned in Section 5.2.2, we focus from now on on geometric double barrier knock-out step call options, i.e. we take $\rho_L, \rho_H \leq 0$ and leverage on the relations obtained in Section 5.2.3. We emphasize however that the approach followed in the upcoming sections is general enough to produce similar results for other types of geometric step contracts and that only few, slight adaptions are needed.

In order to price both European-type as well as American-type double barrier step (call) options, it is sufficient to focus on corresponding step contracts that are initiated at the valuation date under consideration. This clearly follows since for $\bullet \in \{E, A\}$ and any $\mathcal{T}, x, \gamma_L, \gamma_H, r, \delta, K, L, H, \rho_L, \rho_H$, and $\Psi_X(\cdot)$, we have that

$$\mathcal{DSC}_\bullet\big(\mathcal{T}, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) = e^{\rho_L \gamma_L + \rho_H \gamma_H} \cdot \mathcal{DSC}_\bullet\big(\mathcal{T}, x, 0, 0; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big). \tag{5.2.24}$$

Therefore, we assume from now on that an exponential Lévy market, described in terms of its characteristic exponent $\Psi_X(\cdot)$, has been pre-specified and concentrate, for $\bullet \in \{E, A\}$, on geometric step contracts of the form

$$\mathcal{DSC}^{\star}_{\bullet}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \mathcal{DSC}_{\bullet}(\mathcal{T}, x, 0, 0;\, r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)), \tag{5.2.25}$$

with $\boldsymbol{\ell} := (L, H)$ and $\boldsymbol{\rho_\ell} := (\rho_L, \rho_H)$.

5.2.4.1 European-Type Contracts

We first treat European-type contracts and characterize them by means of partial integro-differential equations (PIDEs). This is the content of the next proposition, whose proof is presented in Appendix A (cf. Section 5.6.1).

Proposition 5.1. *For any fixed $T > 0$, strike $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$, the value of the European-type geometric double barrier step call, $\mathcal{DSC}^{\star}_{E}(\cdot)$, is continuous on $[0, T] \times [0, \infty)$ and solves the partial integro-differential equation*

$$-\,\partial_{\mathcal{T}} \mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) + \mathcal{A}_S \mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \left(r - \boldsymbol{\rho_\ell} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix} \right) \mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \tag{5.2.26}$$

on $(0, T] \times [0, \infty)$ with initial condition

$$\mathcal{DSC}^{\star}_{E}(0, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x - K)^{+}, \qquad x \in [0, \infty). \tag{5.2.27}$$

5.2.4.2 American-Type Contracts

We now discuss American-type contracts. First, as in the proof of Proposition 5.1, we note that American-type double barrier step call options can be re-expressed in the form

$$\mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \mathbb{E}^{\mathbb{Q}}_{x} \left[\left(\bar{S}_{\tau} - K \right)^{+} \right], \tag{5.2.28}$$

where $(\bar{S}_t)_{t \geq 0}$ refers to the (strong) Markov process obtained by "killing"[3] the sample path of $(S_t)_{t \geq 0}$ at the proportional rate $\lambda(x) := r - \boldsymbol{\rho_\ell} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix}$ and whose cemetery state is given, without loss of generality, by $\partial \equiv 0$. Therefore, using the fact that the payoff function $x \mapsto (x - K)^{+}$ is continuous as well as standard optimal stopping arguments (cf. Corollary 2.9. and Remark 2.10. in [PS06]), we obtain that the continuation and stopping regions read for a (fixed) valuation horizon $[0, T]$, respectively

$$\mathcal{D}_c = \left\{ (\mathcal{T}, x) \in [0, T] \times [0, \infty) :\, \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) > (x - K)^{+} \right\}, \tag{5.2.29}$$

$$\mathcal{D}_s = \left\{ (\mathcal{T}, x) \in [0, T] \times [0, \infty) :\, \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x - K)^{+} \right\}, \tag{5.2.30}$$

and that, for any $\mathcal{T} \in [0, T]$, the first-entry time

$$\tau_{\mathcal{D}_s} := \inf \left\{ 0 \leq t \leq \mathcal{T} :\, (\mathcal{T} - t, \bar{S}_t) \in \mathcal{D}_s \right\} \tag{5.2.31}$$

is optimal in (5.2.28). This subsequently allows us to make use of standard strong Markovian arguments to derive a characterization of the American-type contract, $\mathcal{DSC}^{\star}_{A}(\cdot)$, in terms of a Cauchy-type problem. This is the content of the next proposition, whose proof is provided in Appendix A (cf. Section 5.6.1).

[3] The reader is referred, for further details, to the proof of Proposition 5.1.

Proposition 5.2. *For any fixed $T > 0$, strike $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$, the value of the American-type geometric double barrier step call, $\mathcal{DSC}_A^\star(\cdot)$, is continuous on $[0,T] \times [0,\infty)$ and satisfies the following Cauchy-type problem:*

$$-\partial_{\mathcal{T}} \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) + \mathcal{A}_S \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \left(r - \boldsymbol{\rho_\ell} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix} \right) \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \tag{5.2.32}$$

for $(\mathcal{T}, x) \in \mathcal{D}_c$ with boundary condition

$$\mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x-K)^+, \qquad \text{for } (\mathcal{T}, x) \in \mathcal{D}_s. \tag{5.2.33}$$

Proposition 5.1 and Proposition 5.2 are of great practical importance since they both provide a characterization of the respective geometric step contracts in terms of a PIDE problem and therefore already allow for a simple treatment of the options $\mathcal{DSC}_E^\star(\cdot)$ and $\mathcal{DSC}_A^\star(\cdot)$ by means of standard numerical techniques. However, these results do not offer any additional insights on the early exercise structure of these options. Instead, an early exercise decomposition into diffusion and jump contributions can be specified and PIDE characterizations thereof can be derived by analyzing the early exercise premium, $\mathcal{E}_{\mathcal{DSC}}^\star(\cdot)$, that is defined, for any $\mathcal{T}, x, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, by

$$\mathcal{E}_{\mathcal{DSC}}^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \mathcal{DSC}_E^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}). \tag{5.2.34}$$

Deriving these characterizations is the content of the following discussion, where we restrict ourselves to jump distributions that are absolutely continuous with respect to the Lebesgue measure, i.e. we only consider Lévy processes whose intensity measure takes the form

$$\Pi_X(dy) = \pi_X(y)dy \tag{5.2.35}$$

for a certain jump density $\pi_X(\cdot)$. This is to ensure that the upcoming decomposition stays meaningful. However, we emphasize that this assumption could be relaxed and additionally note that it does not constitute a real restriction since (almost) all Lévy processes studied in the financial literature satisfy this property.

We start our discussion by noting that the stopping region $\mathcal{D}_s$ is a closed and left-connected[4] set in $[0,T] \times [0,\infty)$ that additionally has the following decomposition

$$\mathcal{D}_s = \mathcal{D}_s^L \cup \mathcal{D}_s^H, \tag{5.2.36}$$

where $\mathcal{D}_s^L$ and $\mathcal{D}_s^H$ are themselves closed and left-connected sets in $[0,T] \times [0,\infty)$, with $\mathcal{D}_s^L$ and $\mathcal{D}_s^H \setminus \{L\}$ being disjoint. This can be seen from the following arguments: First, the closedness of $\mathcal{D}_s$ directly follows from the continuity of the function $(\mathcal{T}, x) \mapsto \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ on $[0,T] \times [0,\infty)$ for any $K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$ (cf. [PS06]), while the fact that $\mathcal{T} \mapsto \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ is, for any $x, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, non-decreasing implies, for $0 \leq \mathcal{T}_1 \leq \mathcal{T}_2 \leq T$, that we have $(\mathcal{T}_1, x) \in \mathcal{D}_s$ whenever $(\mathcal{T}_2, x) \in \mathcal{D}_s$. This already gives what is often referred to as left-connectedness. Therefore, we only have to prove the disjointness of the sets $\mathcal{D}_s^L$, $\mathcal{D}_s^H \setminus \{L\}$ in the decomposition (5.2.36). To see this property, we note that, for any $\mathcal{T}, x, K$, and $\boldsymbol{\ell}$, the following inequality holds

$$\mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \tilde{\boldsymbol{\rho}}_{\boldsymbol{\ell}}) \leq \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}), \qquad \text{whenever } \tilde{\boldsymbol{\rho}}_{\boldsymbol{\ell}} \leq \boldsymbol{\rho_\ell}, \tag{5.2.37}$$

[4] We define left-connectedness in terms of the time to maturity and require the following property:

$$\forall 0 \leq \mathcal{T}_1 \leq \mathcal{T}_2 \leq T,\ x \in [0,\infty): \big((\mathcal{T}_2, x) \in \mathcal{D}_s \Rightarrow (\mathcal{T}_1, x) \in \mathcal{D}_s \big).$$

where $(\tilde{\rho}_L, \tilde{\rho}_H) = \tilde{\boldsymbol{\rho}}_{\boldsymbol{\ell}} \leq \boldsymbol{\rho}_{\boldsymbol{\ell}} = (\rho_L, \rho_H)$ refers to the componentwise inequalities $\tilde{\rho}_L \leq \rho_L$ and $\tilde{\rho}_H \leq \rho_H$. Since standard options are recovered from geometric double barrier step options by replacing $\boldsymbol{\rho}_{\boldsymbol{\ell}}$ with $\boldsymbol{\rho}_{\boldsymbol{\ell}}^{\boldsymbol{S}} := (0,0)$ in (5.2.25) and (standard) double barrier knock-out options can be understood as "limit" of geometric double barrier step contracts, e.g. via the sequence $\big(\boldsymbol{\rho}_{n,\boldsymbol{\ell}}^{\boldsymbol{B}}\big)_{n\in\mathbb{N}} := \big((-n,-n)\big)_{n\in\mathbb{N}}$, we obtain, in particular, that

$$\mathcal{DBC}_A(\mathcal{T}, x; K, \boldsymbol{\ell}) \leq \mathcal{DSC}_A^{\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho}_{\boldsymbol{\ell}}) \leq \mathcal{C}_A(\mathcal{T}, x; K). \tag{5.2.38}$$

Here, $\mathcal{C}_A(\cdot)$ and $\mathcal{DBC}_A(\cdot)$ refer to the (standard) American-type call and the (standard) American-type double barrier knock-out call, obtained by

$$\mathcal{C}_A(\mathcal{T}, x; K) := \mathcal{DSC}_A^{\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho}_{\boldsymbol{\ell}}^{\boldsymbol{S}}), \tag{5.2.39}$$

$$\mathcal{DBC}_A(\mathcal{T}, x; K, \boldsymbol{\ell}) := \sup_{\tau \in \mathfrak{T}_{[0,\mathcal{T}]}} \lim_{n\uparrow\infty} \mathcal{DSC}^{\star}(\tau, x; K, \boldsymbol{\ell}, \boldsymbol{\rho}_{n,\boldsymbol{\ell}}^{\boldsymbol{B}}), \tag{5.2.40}$$

where $\mathcal{DSC}^{\star}(\tau, x; K, \boldsymbol{\ell}, \boldsymbol{\rho}_{\boldsymbol{\ell}}) = \mathcal{DSC}\big(\tau, x, 0, 0;\, r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big)$ denotes the version (5.2.14) that is initiated at the valuation date under consideration, i.e. in the sense of the notation introduced in (5.2.25). Hence, this gives that $\mathcal{D}_{S,s} \subseteq \mathcal{D}_s \subseteq \mathcal{D}_{B,s}$, with $\mathcal{D}_{S,s}$ and $\mathcal{D}_{B,s}$ denoting the stopping region of the corresponding (standard) American-type call and (standard) American-type double barrier knock-out call, respectively, i.e.

$$\mathcal{D}_{S,s} = \big\{(\mathcal{T}, x) \in [0,T] \times [0,\infty) :\, \mathcal{C}_A(\mathcal{T}, x; K) = (x-K)^+\big\}, \tag{5.2.41}$$

$$\mathcal{D}_{B,s} = \big\{(\mathcal{T}, x) \in [0,T] \times [0,\infty) :\, \mathcal{DBC}_A(\mathcal{T}, x; K, \boldsymbol{\ell}) = (x-K)^+\big\}. \tag{5.2.42}$$

In particular, $\mathcal{D}_{S,s} \subseteq \mathcal{D}_s$ directly implies, for $\delta > 0$, the non-emptyness of the stopping region $\mathcal{D}_s$ (cf. [Ma20]), whereas combining well-known results for (standard) American-type double barrier options with the relation $\mathcal{D}_s \subseteq \mathcal{D}_{B,s}$ gives that early exercise of the geometric double barrier knock-out step call can only occur, for a fixed $\mathcal{T} \in [0,T]$, in subregions of the intervals $I_1 := (K, L]$, whenever $L > K$, and $I_2 := \big[\mathfrak{b}^B(\mathcal{T}), \infty\big)$, where $\mathfrak{b}^B(\mathcal{T}) \geq \max(K, L)$ denotes the early exercise up-boundary of the corresponding (standard) American-type double barrier knock-out call. This provides (5.2.36).

Next, combining the closedness of $\mathcal{D}_s$ with its left-connectedness and decomposition (5.2.36) leads to the following observations:[5] First, any entry of the stopping region that is triggered by the diffusion part of the process $(S_t)_{t\geq 0}$[6] will happen by crossing the boundary $\partial\mathcal{D}_s$ of the set $\mathcal{D}_s$, where

$$\partial\mathcal{D}_s := \Big\{(\mathcal{T}, x) \in \mathcal{D}_s :\, \forall \epsilon > 0 : B_\epsilon\big((\mathcal{T}, x)\big) \cap \mathcal{D}_s \neq \emptyset \;\wedge\; B_\epsilon\big(((\mathcal{T}, x)\big) \cap \Big(\big([0,T] \times [0,\infty)\big) \setminus \mathcal{D}_s\Big) \neq \emptyset\Big\},$$

and $B_\epsilon\big((\mathcal{T}, x)\big)$ denotes the open ball around the (mid-)point $(\mathcal{T}, x)$ and with radius $\epsilon > 0$. On the other hand, first-passage entries in the stopping region that are triggered by jumps will always occur at an interior point of the set $\mathcal{D}_s$, i.e. within $\mathcal{D}_s^{\circ} := \mathcal{D}_s \setminus \partial\mathcal{D}_s$, whenever the $\mathcal{T}$-section $\mathcal{D}_{s,\mathcal{T}} := \{x \in [0,\infty) :\, (\mathcal{T}, x) \in \mathcal{D}_s\}$ contains, for all $\mathcal{T} \in [0,T]$, only finitely many x with $(\mathcal{T}, x) \in \partial\mathcal{D}_s$, i.e. whenever we have for all $\mathcal{T} \in [0,T]$ that $\#\big(\partial\mathcal{D}_s \cap \big(\{\mathcal{T}\} \times \mathcal{D}_{s,\mathcal{T}}\big)\big) < \infty$. This is a direct consequence of Assumption (5.2.35), as this assumption implies that, conditional on a jump occuring at time t, events of the form $\{S_t = \varphi + S_{t-}\}$ have for any fixed $\varphi \in \mathbb{R}$ zero probability. Additionally, in cases where $\#\big(\partial\mathcal{D}_s \cap \big(\{\mathcal{T}_0\} \times \mathcal{D}_{s,\mathcal{T}_0}\big)\big) = \infty$ holds for some $\mathcal{T}_0 \in [0,T]$, the stopping region has the particularity to suddenly increase in size at this particular point in time $\mathcal{T}_0$ and any entry in $\partial\mathcal{D}_s \cap \big(\{\mathcal{T}_0\} \times \mathcal{D}_{s,\mathcal{T}_0}\big)$ is very much due to the drastic change in the shape of the stopping region at this point. In particular, since Lévy processes are quasi left-continuous, i.e. left-continuous

[5] We refer the reader for similar ideas to [FMV19]; see also [LV17] and [CV18].

[6] Or, equivalently, by the diffusion part of the underlying Lévy process $(X_t)_{t\geq 0}$.

over predictable stopping times, these stopping scenarios can only be due to the diffusion part of the process $(S_t)_{t\geq 0}$. Consequently, these observations justify the usage of the sets $\partial\mathcal{D}_s$ and $\mathcal{D}_s^{\circ}$ to decompose the stopping region $\mathcal{D}_s$ into sub-regions where stopping is purely triggered by diffusion and by jumps, respectively. This subsequently results in a decomposition of the early exercise premium, $\mathcal{E}^{\star}_{\mathcal{DSC}}(\cdot)$, of the following form:

$$\mathcal{E}^{\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) + \mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}). \tag{5.2.43}$$

Here, the premiums $\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\cdot)$ and $\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\cdot)$ refer to the early exercise contributions of the diffusion and jump parts, respectively, and are defined in the following way

$$\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) := \mathcal{DSC}^{0,\star}_{A}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \mathcal{DSC}^{0,\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}), \tag{5.2.44}$$

$$\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) := \mathcal{DSC}^{\mathcal{J},\star}_{A}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \mathcal{DSC}^{\mathcal{J},\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}), \tag{5.2.45}$$

where the European-type functions $\mathcal{DSC}^{0,\star}_{E}(\cdot)$ and $\mathcal{DSC}^{\mathcal{J},\star}_{E}(\cdot)$ are given by

$$\mathcal{DSC}^{0,\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\mathcal{T}}-K\right)^{+}\mathbb{1}_{\partial\mathcal{D}_s}\big((\mathcal{T}-\tau_{\mathcal{D}_s},\bar{S}_{\tau_{\mathcal{D}_s}})\big)\right], \tag{5.2.46}$$

$$\mathcal{DSC}^{\mathcal{J},\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\mathcal{T}}-K\right)^{+}\mathbb{1}_{\mathcal{D}_s^{\circ}}\big((\mathcal{T}-\tau_{\mathcal{D}_s},\bar{S}_{\tau_{\mathcal{D}_s}})\big)\right], \tag{5.2.47}$$

and the American-type contributions $\mathcal{DSC}^{0,\star}_{A}(\cdot)$ and $\mathcal{DSC}^{\mathcal{J},\star}_{A}(\cdot)$ are defined, accordingly, as

$$\mathcal{DSC}^{0,\star}_{A}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\tau_{\mathcal{D}_s}}-K\right)^{+}\mathbb{1}_{\partial\mathcal{D}_s}\big((\mathcal{T}-\tau_{\mathcal{D}_s},\bar{S}_{\tau_{\mathcal{D}_s}})\big)\right], \tag{5.2.48}$$

$$\mathcal{DSC}^{\mathcal{J},\star}_{A}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\tau_{\mathcal{D}_s}}-K\right)^{+}\mathbb{1}_{\mathcal{D}_s^{\circ}}\big((\mathcal{T}-\tau_{\mathcal{D}_s},\bar{S}_{\tau_{\mathcal{D}_s}})\big)\right]. \tag{5.2.49}$$

Combining these definitions with strong Markovian arguments finally allows us to derive PIDE characterizations of the early exercise contributions $\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\cdot)$ and $\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\cdot)$. This is the content of the next proposition, whose proof is presented in Appendix A (cf. Section 5.6.1).

Proposition 5.3. *For any fixed $T>0$, strike $K\geq 0$, barrier levels $0\leq L\leq H<\infty$, and knock-out rates $\rho_L,\rho_H\leq 0$, the value of the diffusion contribution to the early exercise premium of the geometric double barrier step call, $\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\cdot)$, satisfies the following Cauchy-type problem:*

$$-\partial_{\mathcal{T}}\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) + \mathcal{A}_S\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \left(r-\boldsymbol{\rho_\ell}\cdot\begin{pmatrix}\mathbb{1}_{(0,L)}(x)\\ \mathbb{1}_{(H,\infty)}(x)\end{pmatrix}\right)\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = 0, \tag{5.2.50}$$

for $(\mathcal{T},x)\in\mathcal{D}_c$ with boundary conditions

$$\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = (x-K)^{+} - \mathcal{DSC}^{\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}), \qquad \text{for } (\mathcal{T},x)\in\partial\mathcal{D}_s, \tag{5.2.51}$$

$$\mathcal{E}^{0,\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = 0, \qquad \text{for } (\mathcal{T},x)\in\mathcal{D}_s^{\circ}. \tag{5.2.52}$$

Similarly, the value of the jump contribution to the early exercise premium of the geometric double barrier step call, $\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\cdot)$, solves the following Cauchy-type problem:

$$-\partial_{\mathcal{T}}\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) + \mathcal{A}_S\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \left(r-\boldsymbol{\rho_\ell}\cdot\begin{pmatrix}\mathbb{1}_{(0,L)}(x)\\ \mathbb{1}_{(H,\infty)}(x)\end{pmatrix}\right)\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = 0, \tag{5.2.53}$$

for $(\mathcal{T},x)\in\mathcal{D}_c$ with boundary conditions

$$\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = 0, \qquad \text{for } (\mathcal{T},x)\in\partial\mathcal{D}_s, \tag{5.2.54}$$

$$\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = (x-K)^{+} - \mathcal{DSC}^{\star}_{E}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}), \qquad \text{for } (\mathcal{T},x)\in\mathcal{D}_s^{\circ}. \tag{5.2.55}$$

Remark 5.2.

Although Proposition 5.3 provides a meaningful characterization of diffusion and jump contributions to the early exercise premium of geometric step options, one may have the impression that these results are lacking applicability. In particular, it seems difficult to make use of these characterizations in practice since the sets $\mathcal{D}_s$, $\partial\mathcal{D}_s$, and $\mathcal{D}_s^\circ$ are usually not known in advance. However, we will see that Proposition 5.3 and the upcoming results of Section 5.2.5 will play a crucial role in Section 5.3, where they will allow for a derivation of semi-analytical diffusion and jump contributions to the early exercise premium of geometric down-and-out step call options under hyper-exponential jump-diffusion markets.

♦

5.2.5 Maturity-Randomization and OIDEs

We next deal with maturity-randomized geometric step contracts. To this end, we consider for a function $g : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\int_0^\infty e^{-\vartheta t} |g(t)| \, dt < \infty, \qquad \forall \vartheta > 0, \tag{5.2.56}$$

the Laplace-Carson transform $\widehat{g}(\cdot)$ defined via

$$\widehat{g}(\vartheta) := \int_0^\infty \vartheta e^{-\vartheta t} \, g(t) \, dt \tag{5.2.57}$$

and note that this transform has several desirable properties.[7] In particular, applying the Laplace-Carson transform in the context of mathematical finance allows to randomize the maturity of (certain) financial contracts, i.e. to switch from objects with deterministic maturity to corresponding objects with stochastic maturity. This last property offers various approaches to the valuation of financial positions and has therefore led to a wide adoption of the Laplace-Carson transform in the option pricing literature, with [Ca98] being one of the seminal articles in this context.

Once an (analytical or numerical) expression for the Laplace-Carson transform has been obtained, inversion is carried out numerically through an inversion algorithm. One possible choice is the Gaver-Stehfest algorithm that has the advantage to allow for an inversion of the transform on the real line and that has been successfully used by several authors in the option pricing literature (cf. [KW03], [Ki10], [WZ10], [HM13], [LV17], [CV18], [LV20]). We will also rely on this algorithm, i.e. we set

$$g_N(t) := \sum_{k=1}^{2N} \zeta_{k,N} \, \mathcal{LC}(g) \left(\frac{k \log(2)}{t} \right), \qquad N \in \mathbb{N}, \; t > 0, \tag{5.2.58}$$

where the coefficients are given by

$$\zeta_{k,N} := \frac{(-1)^{N+k}}{k} \sum_{j=\lfloor (k+1)/2 \rfloor}^{\min\{k,N\}} \frac{j^{N+1}}{N!} \binom{N}{j} \binom{2j}{j} \binom{j}{k-j}, \qquad N \in \mathbb{N}, \; 1 \le k \le 2N, \tag{5.2.59}$$

[7] We refer the interested reader to [KW03], [Ki10], [LV17], and [FMV19] for a discussion of some of these properties.

with $\lfloor a \rfloor := \sup\{z \in \mathbb{Z} : \, z \leq a\}$, and will recover the original function $g(\cdot)$ by means of the following relation

$$\lim_{N \to \infty} g_N(t) = g(t). \tag{5.2.60}$$

More technical details around the Gaver-Stehfest inversion as well as formal proofs of the convergence result (5.2.60) for "sufficiently well-behaved functions" are provided in [VA04], [AW06], [Ku13], and references therein.

5.2.5.1 European-Type Contracts

To start, we focus on maturity-randomized versions of the European-type geometric step option $\mathcal{DSC}^{\star}_{E}(\cdot)$, i.e. we consider geometric step contracts of the form

$$\widehat{\mathcal{DSC}^{\star}_{E}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) := \mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\mathcal{T}_{\vartheta}} - K \right)^{+} \right], \tag{5.2.61}$$

where $(\bar{S}_t)_{t\geq 0}$ refers, once again, to the (strong) Markov process obtained by "killing" the sample path of $(S_t)_{t\geq 0}$ at the proportional rate $\lambda(x) := r - \boldsymbol{\rho_{\ell}} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix}$ and whose cemetery state is given by $\partial \equiv 0$, and $\mathcal{T}_{\vartheta}$ denotes an exponentially distributed random time of intensity $\vartheta > 0$ that is independent of $(S_t)_{t\geq 0}$. It is not hard to see that (5.2.61) re-writes as

$$\widehat{\mathcal{DSC}^{\star}_{E}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) = \mathbb{E}^{\mathbb{Q}}_{x}\left[\mathbb{E}^{\mathbb{Q}}_{x}\left[\left(\bar{S}_{\mathcal{T}_{\vartheta}} - K \right)^{+} \middle| \mathcal{T}_{\vartheta} \right] \right] = \int_{0}^{\infty} \vartheta e^{-\vartheta t} \, \mathcal{DSC}^{\star}_{E}(t, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) \, dt, \tag{5.2.62}$$

and therefore that the maturity-randomized versions (5.2.61) correspond, for any fixed $x, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_{\ell}}$, to a strict application of the Laplace-Carson transform to the function $\mathcal{T} \mapsto \mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}})$. Additionally, we note that this transform is well-defined. Indeed, this was already shown in a slightly different context for standard (European- and American-type) options in [Ma20] and directly follows from these results, for $\boldsymbol{\rho_{\ell}} \leq 0$ and $\bullet \in \{E, A\}$, by means of the inequality

$$\mathcal{DSC}^{\star}_{\bullet}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) \leq \mathcal{DSC}^{\star}_{\bullet}(\mathcal{T}, x; K, \boldsymbol{\ell}, (0,0)) =: \mathcal{C}_{\bullet}(\mathcal{T}, x; K). \tag{5.2.63}$$

Consequently, combining these properties with arguments similarly used in the proof of Proposition 5.1 allows to obtain an OIDE characterization of the maturity-randomized European-type contracts (5.2.61). This is the content of the next proposition, whose proof is provided in Appendix A (cf. Section 5.6.1).

Proposition 5.4. *For any intensity $\vartheta > 0$, strike $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$, the value of the maturity-randomized European-type geometric double barrier step call, $\widehat{\mathcal{DSC}^{\star}_{E}}(\cdot)$, is continuous on $[0, \infty)$ and solves the ordinary integro-differential equation*

$$\vartheta(x-K)^{+} + \mathcal{A}_S \widehat{\mathcal{DSC}^{\star}_{E}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) - \left((r+\vartheta) - \boldsymbol{\rho_{\ell}} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix} \right) \widehat{\mathcal{DSC}^{\star}_{E}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) = 0, \tag{5.2.64}$$

on $(0, \infty)$ with initial condition

$$\widehat{\mathcal{DSC}^{\star}_{E}}(\vartheta, 0; K, \boldsymbol{\ell}, \boldsymbol{\rho_{\ell}}) = 0. \tag{5.2.65}$$

5.2.5.2 American-Type Contracts

Lastly, we discuss maturity-randomized versions of the American-type geometric step option $\mathcal{DSC}_A^{\star}(\cdot)$, i.e. we consider the following geometric step contracts

$$\widehat{\mathcal{DSC}_A^{\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau} - K\right)^+\right], \tag{5.2.66}$$

where we use the notation introduced in Section 5.2.5.1. Due to their complex early exercise structure, these maturity-randomized contracts do not anymore coincide with a strict application of the Laplace-Carson transform to their deterministic counterparts $\mathcal{T} \mapsto \mathcal{DSC}_A^{\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$. Instead, conditioning on the (independent) exponential random time $\mathcal{T}_\vartheta$ only leads to the following expression

$$\widehat{\mathcal{DSC}_A^{\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathbb{E}_x^{\mathbb{Q}}\left[\mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau} - K\right)^+ \big| \mathcal{T}_\vartheta\right]\right] = \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \int_0^\infty \vartheta e^{-\vartheta t}\, \mathcal{DSC}^{\star}(t \wedge \tau, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})\, dt, \tag{5.2.67}$$

where $\mathcal{DSC}^{\star}(\tau, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathcal{DSC}\big(\tau, x, 0, 0;\, r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big)$ denotes, as earlier, for any $T > 0$ and stopping time $\tau \in \mathfrak{T}_{[0,T]}$, the contract version of (5.2.14) that is initiated at the valuation date under consideration, i.e. in the sense of the notation introduced in (5.2.25). Nevertheless, the same arguments as in Section 5.2.5.1 (cf. [Ma20]) directly show that the right-hand side in (5.2.67) is well-defined for $\boldsymbol{\rho_\ell} \leq 0$ and any $\vartheta > 0$. Furthermore, OIDE characterizations of the maturity-randomized American-type contract $\widehat{\mathcal{DSC}_A^{\star}}(\cdot)$ as well as of the respective early exercise premiums can be derived using strong Markovian arguments. This is the content of the following discussion.

To start, we recall that the (independent) exponential random time $\mathcal{T}_\vartheta$ can be interpreted as the (first) jump time of a corresponding (independent) Poisson process $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$ and that this can be used to re-express the optimal stopping problem in a slightly different form. In particular, we can consider, for any $\vartheta > 0$ and initial value $z = (n, x) \in \mathbb{N}_0 \times [0, \infty)$, the process $(Z_t)_{t\geq 0}$ defined on the state domain $\mathcal{D} := \mathbb{N}_0 \times [0,\infty)$ via $Z_t := (n + N_t, \bar{S}_t)$, $\bar{S}_0 = x$, as well as its stopped version, $(Z_t^{\mathcal{S}_J})_{t\geq 0}$, defined, for $t \geq 0$, via

$$Z_t^{\mathcal{S}_J} := Z_{t \wedge \tau_{\mathcal{S}_J}}, \qquad \text{with} \qquad \tau_{\mathcal{S}_J} := \inf\{t \geq 0 :\, Z_t \in \mathcal{S}_J\}, \qquad \text{and} \qquad \mathcal{S}_J := \mathbb{N} \times [0,\infty). \tag{5.2.68}$$

Clearly, the process $(Z_t^{\mathcal{S}_J})_{t\geq 0}$ behaves exactly like the process $(Z_t)_{t\geq 0}$ for all times $t < \tau_{\mathcal{S}_J}$, which implies that most of the properties of $(Z_t)_{t\geq 0}$ naturally extend to $(Z_t^{\mathcal{S}_J})_{t\geq 0}$.[8] Additionally, $\widehat{\mathcal{DSC}_A^{\star}}(\cdot)$ can be re-expressed, for $\vartheta, K, \boldsymbol{\ell}$ and $\boldsymbol{\rho_\ell}$, in the form

$$\widehat{\mathcal{DSC}_A^{\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \widehat{V_A}\big((0,x)\big), \tag{5.2.69}$$

where the value function $\widehat{V_A}(\cdot)$ has the following representation under the measure $\mathbb{Q}_z^Z$ having initial distribution $Z_0 = z$:

$$\widehat{V_A}(z) := \sup_{\tau \in \mathfrak{T}_{[0,\infty)}} \mathbb{E}_z^{\mathbb{Q}^Z}\big[G(Z_\tau^{\mathcal{S}_J})\big], \qquad G(z) := (x - K)^+. \tag{5.2.70}$$

Therefore, using the fact that the payoff function $x \mapsto (x-K)^+$ is continuous as well as standard optimal stopping arguments (cf. Corollary 2.9. and Remark 2.10. in [PS06]), we can infer that the continuation and stopping regions to (the more general) Problem (5.2.70) read, respectively

$$\widehat{\mathcal{D}_c^{Gen.}} = \Big\{z \in \mathcal{D} :\, \widehat{V_A}(z) > G(z)\Big\}, \qquad \text{and} \qquad \widehat{\mathcal{D}_s^{Gen.}} = \Big\{z \in \mathcal{D} :\, \widehat{V_A}(z) = G(z)\Big\}, \tag{5.2.71}$$

[8] In particular, the process $(Z_t^{\mathcal{S}_J})_{t\geq 0}$ is again strongly Markovian on the state domain $\mathcal{D}$.

and that the first-entry time

$$\tau_{\widehat{\mathcal{D}_s^{Gen.}}} := \inf\left\{t \geq 0 :\, Z_t^{\mathcal{S}_J} \in \widehat{\mathcal{D}_s^{Gen.}}\right\} \tag{5.2.72}$$

is optimal in (5.2.70).[9] This then allows us to make use of standard strong Markovian arguments to derive a characterization of the American-type contract $\widehat{\mathcal{DSC}_A^\star}(\cdot)$ in terms of a Cauchy-type problem and leads via Relation (5.2.69) and the following continuation and stopping regions

$$\widehat{\mathcal{D}}_{\vartheta,c} = \left\{x \in [0,\infty) :\, \widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) > (x-K)^+\right\}, \tag{5.2.73}$$

$$\widehat{\mathcal{D}}_{\vartheta,s} = \left\{x \in [0,\infty) :\, \widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x-K)^+\right\}, \tag{5.2.74}$$

to the next proposition. A proof is presented in Appendix A (cf. Section 5.6.1).

Proposition 5.5. *For any intensity $\vartheta > 0$, strike $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$, the value of the maturity-randomized American-type geometric double barrier step call, $\widehat{\mathcal{DSC}_A^\star}(\cdot)$, is continuous on $[0,\infty)$ and satisfies the following Cauchy-type problem:*

$$\vartheta(x-K)^+ + \mathcal{A}_S\widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \left((r+\vartheta) - \boldsymbol{\rho_\ell}\cdot\begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x)\end{pmatrix}\right)\widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \tag{5.2.75}$$

for $x \in \widehat{\mathcal{D}}_{\vartheta,c}$ with boundary condition

$$\widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x-K)^+, \qquad \text{for } x \in \widehat{\mathcal{D}}_{\vartheta,s}. \tag{5.2.76}$$

To finalize our discussion, we aim to characterize diffusion and jump contributions to the maturity-randomized early exercise premium of geometric double barrier step contracts, that is defined for $\vartheta, x, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$ via

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \widehat{\mathcal{DSC}_E^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}). \tag{5.2.77}$$

For simplicity of the exposition, we directly rely on the continuation and stopping regions introduced in (5.2.73), (5.2.74) and note that the maturity-randomized American-type option $\widehat{\mathcal{DSC}_A^\star}(\cdot)$ can be equivalently written as

$$\widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}} - K\right)^+\right], \tag{5.2.78}$$

since the first-entry time $\tau_{\widehat{\mathcal{D}}_{\vartheta,s}} := \inf\left\{t \geq 0 :\, \bar{S}_t \in \widehat{\mathcal{D}}_{\vartheta,s}\right\}$ clearly inherits the optimality of its counterpart (5.2.72) in the more general problem (5.2.70). Then, following the line of the arguments provided in Section 5.2.4.2, we can make use of the sets $\partial\widehat{\mathcal{D}}_{\vartheta,s}$ and $\widehat{\mathcal{D}}_{\vartheta,s}^\circ$ to decompose the stopping region into subregions where (early) stopping is purely due to diffusion and jumps, respectively, and subsequently derive a decomposition of the maturity-randomized early exercise premium, $\widehat{\mathcal{E}_{\mathcal{DSC}}^\star}(\cdot)$, of the form

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) + \widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}). \tag{5.2.79}$$

Here, the premiums $\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\cdot)$ and $\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\cdot)$ refer to the maturity-randomized early exercise contributions of the diffusion and jump parts, respectively, and are defined via

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \widehat{\mathcal{DSC}_A^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \widehat{\mathcal{DSC}_E^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}), \tag{5.2.80}$$

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) := \widehat{\mathcal{DSC}_A^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \widehat{\mathcal{DSC}_E^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}), \tag{5.2.81}$$

[9] Note that the finiteness of this stopping time directly follows from the finiteness of the first moment of any exponential distribution and the fact that $\mathcal{S}_J \subseteq \widehat{\mathcal{D}_s^{Gen.}}$.

where the maturity-randomized European-type functions $\widehat{\mathcal{DSC}}_E^{0,\star}(\cdot)$ and $\widehat{\mathcal{DSC}}_E^{\mathcal{J},\star}(\cdot)$ are given by

$$\widehat{\mathcal{DSC}}_E^{0,\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta} - K\right)^+ \mathbb{1}_{\partial\widehat{\mathcal{D}}_{\vartheta,s}}\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}}\right)\right], \tag{5.2.82}$$

$$\widehat{\mathcal{DSC}}_E^{\mathcal{J},\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta} - K\right)^+ \mathbb{1}_{\widehat{\mathcal{D}}_{\vartheta,s}^{\circ}}\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}}\right)\right], \tag{5.2.83}$$

and the maturity-randomized American-type contributions $\widehat{\mathcal{DSC}}_A^{0,\star}(\cdot)$ and $\widehat{\mathcal{DSC}}_A^{\mathcal{J},\star}(\cdot)$ are defined accordingly, as

$$\widehat{\mathcal{DSC}}_A^{0,\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}} - K\right)^+ \mathbb{1}_{\partial\widehat{\mathcal{D}}_{\vartheta,s}}\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}}\right)\right], \tag{5.2.84}$$

$$\widehat{\mathcal{DSC}}_A^{\mathcal{J},\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \mathbb{E}_x^{\mathbb{Q}}\left[\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}} - K\right)^+ \mathbb{1}_{\widehat{\mathcal{D}}_{\vartheta,s}^{\circ}}\left(\bar{S}_{\mathcal{T}_\vartheta \wedge \tau_{\widehat{\mathcal{D}}_{\vartheta,s}}}\right)\right]. \tag{5.2.85}$$

Combining these definitions with strong Markovian arguments similarly used in the proof of the previous propositions and the memorylessness of the exponential distribution finally allows us to derive OIDE characterizations of the early exercise contributions $\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\cdot)$ and $\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\cdot)$. This is the content of the next proposition, whose proof is provided in Appendix A (cf. Section 5.6.1).

Proposition 5.6. *For any intensity $\vartheta > 0$, strike $K \geq 0$, barrier levels $0 \leq L \leq H < \infty$, and knock-out rates $\rho_L, \rho_H \leq 0$, the value of the diffusion contribution to the maturity-randomized early exercise premium of the geometric double barrier step call, $\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\cdot)$, satisfies the following Cauchy-type problem:*

$$\mathcal{A}_S\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \left((r+\vartheta) - \boldsymbol{\rho_\ell} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix}\right)\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \tag{5.2.86}$$

for $x \in \widehat{\mathcal{D}}_{\vartheta,c}$ with boundary conditions

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x-K)^+ - \widehat{\mathcal{DSC}}_E^{\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}), \qquad \text{for } x \in \partial\widehat{\mathcal{D}}_{\vartheta,s}, \tag{5.2.87}$$

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{0,\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \qquad \text{for } x \in \widehat{\mathcal{D}}_{\vartheta,s}^{\circ}. \tag{5.2.88}$$

Similarly, the value of the jump contribution to the maturity-randomized early exercise premium of the geometric double barrier step call, $\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\cdot)$, solves the following Cauchy-type problem:

$$\mathcal{A}_S\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \left((r+\vartheta) - \boldsymbol{\rho_\ell} \cdot \begin{pmatrix} \mathbb{1}_{(0,L)}(x) \\ \mathbb{1}_{(H,\infty)}(x) \end{pmatrix}\right)\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \tag{5.2.89}$$

for $x \in \widehat{\mathcal{D}}_{\vartheta,c}$ with boundary conditions

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = 0, \qquad \text{for } x \in \partial\widehat{\mathcal{D}}_{\vartheta,s}, \tag{5.2.90}$$

$$\widehat{\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = (x-K)^+ - \widehat{\mathcal{DSC}}_E^{\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}), \qquad \text{for } x \in \widehat{\mathcal{D}}_{\vartheta,c}^{\circ}. \tag{5.2.91}$$

<u>**Remark 5.3.**</u>
Although maturity-randomized American-type contracts and maturity-randomized early exercise premiums do not anymore coincide with a strict application of the Laplace-Carson transform to their deterministic counterparts, they exhibit a very similar structure. This becomes clear when comparing Equations (5.2.67) and (5.2.77) with Identity (5.2.62). Hence, once (analytical or numerical) results are obtained for these quantities, a very natural pricing algorithm consists in dealing with their results as if they would actually correspond to proper Laplace-Carson applications and therefore in inverting them via an algorithm such as the one proposed in the Gaver-Stehfest inversion. This has been already investigated by other authors in a similar context (cf. [WZ10], [LV17], [CV18]) where this approach has proven to deliver a very good pricing accuracy. We will follow the idea of this literature and will provide in Section 5.4 numerical results for geometric down-and-out step call options under hyper-exponential jump-diffusion markets based on this ansatz. This also justifies our slight abuse of notation in the current section, where we intentionally used for both maturity-randomized American-type contracts as well as maturity-randomized early exercise premiums the same notation as for Laplace-Carson transforms.

♦

5.3 Geometric Step Options and Hyper-Exponential Jump-Diffusion Markets

As an application of the theory developed in Section 5.2, we derive semi-analytical pricing results for (regular) geometric down-and-out step call options under hyper-exponential jump-diffusion markets, i.e. we fix in (5.2.1), (5.2.2) hyper-exponential jump-diffusion dynamics $(X_t)_{t\geq 0}$ and consider geometric step options of the form

$$\mathcal{DOSC}_{\bullet}^{\star}(\mathcal{T},x;K,L,\rho_L) := \mathcal{DSC}_{\bullet}^{\star}\big(\mathcal{T},x;K,(L,L),(\rho_L,0)\big), \tag{5.3.1}$$

for $\bullet \in \{E,A\}$, time to maturity $\mathcal{T}\geq 0$, initial value $x\geq 0$, strike $K\geq 0$, lower barrier $0\leq L\leq K<\infty$ and knock-out rate $\rho_L\leq 0$.

5.3.1 Generalities on Hyper-Exponential Jump-Diffusion Markets

We recall that a hyper-exponential jump-diffusion market is a Lévy market consisting of a deterministic savings account $(B_t(r))_{t\geq 0}$ (cf. (5.2.1)) and a risky asset $(S_t)_{t\geq 0}$ (cf. (5.2.2)) whose driving process $(X_t)_{t\geq 0}$ combines a Brownian diffusion with hyper-exponentially distributed jumps. In particular, the underlying dynamics $(X_t)_{t\geq 0}$ have the usual jump-diffusion structure, i.e. they can be characterized on a filtered probability space $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P})$ via

$$X_t = \Big(r-\delta-\lambda\zeta-\frac{1}{2}\sigma_X^2\Big)t + \sigma_X W_t + \sum_{i=1}^{N_t} J_i, \qquad t\geq 0, \tag{5.3.2}$$

where $(W_t)_{t\geq 0}$ denotes an $\mathbf{F}$-Brownian motion and $(N_t)_{t\geq 0}$ is an $\mathbf{F}$-Poisson process having intensity parameter $\lambda>0$. The constants $\zeta := \mathbb{E}^{\mathbb{Q}}\left[e^{J_1}-1\right]$ and $\sigma_X>0$ express the average (percentage) jump size and the volatility of the diffusion part, respectively. Additionally, the jumps $(J_i)_{i\in\mathbb{N}}$ are assumed to be independent of $(N_t)_{t\geq 0}$ and to form a sequence of independent and identically distributed random variables following a hyper-exponential distribution, i.e. their (common) density function $f_{J_1}(\cdot)$ is given by

$$f_{J_1}(y) = \sum_{i=1}^{m} p_i\xi_i e^{-\xi_i y}\mathbb{1}_{\{y\geq 0\}} + \sum_{j=1}^{n} q_j\eta_j e^{\eta_j y}\mathbb{1}_{\{y<0\}}, \tag{5.3.3}$$

where $p_i > 0$ and $\xi_i > 1$ for $i \in \{1, \dots, m\}$ and $q_j > 0$ and $\eta_j > 0$ for $j \in \{1, \dots, n\}$. Here, the parameters $(p_i)_{i\in\{1,\dots,m\}}$ and $(q_j)_{j\in\{1,\dots,n\}}$ represent the proportion of jumps that are attributed to particular jump types and are therefore assumed to satisfy the condition $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. For notational simplicity, we require that the intensity parameters $(\xi_i)_{i\in\{1,\dots,m\}}$ and $(\eta_j)_{j\in\{1,\dots,n\}}$ are ordered in the sense that

$$\xi_1 < \xi_2 < \cdots < \xi_m \qquad \text{and} \qquad \eta_1 < \eta_2 < \cdots < \eta_n \tag{5.3.4}$$

and note that this does not consist in a loss of generality.

As special class of Lévy markets, hyper-exponential jump-diffusion markets can be equivalently characterized in terms of their Lévy triplet $\big(b_X, \sigma_X^2, \Pi_X\big)$, where b_X and Π_X are then obtained as

$$b_X := \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma_X^2\Big) + \int_{\{|y|\le 1\}} y\,\Pi_X(dy) \qquad \text{and} \qquad \Pi_X(dy) := \lambda f_{J_1}(y)dy. \tag{5.3.5}$$

Combining these results with Equation (5.2.3), their Lévy exponent, $\Psi_X(\cdot)$, is then easily derived as

$$\Psi_X(\theta) = -i\Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma_X^2\Big)\theta + \frac{1}{2}\sigma_X^2\theta^2 - \lambda\left(\sum_{i=1}^{m}\frac{p_i\xi_i}{\xi_i - i\theta} + \sum_{j=1}^{n}\frac{q_j\eta_j}{\eta_j + i\theta} - 1\right). \tag{5.3.6}$$

Similarly, their Laplace exponent, $\Phi_X(\cdot)$, is well-defined for $\theta \in (-\eta_1, \xi_1)$ and equals

$$\Phi_X(\theta) = \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma_X^2\Big)\theta + \frac{1}{2}\sigma_X^2\theta^2 + \lambda\left(\sum_{i=1}^{m}\frac{p_i\xi_i}{\xi_i - \theta} + \sum_{j=1}^{n}\frac{q_j\eta_j}{\eta_j + \theta} - 1\right). \tag{5.3.7}$$

In what follows, we will consider the Laplace exponent as standalone function on the extended real domain $\Phi_X : \mathbb{R}\setminus\{\xi_1, \dots, \xi_m, -\eta_1, \dots, -\eta_n\} \to \mathbb{R}$. This quantity will play a central role in the upcoming derivations. In fact, many distributional properties of hyper-exponential jump-diffusion markets (and of their generalizations) are closely linked to the roots of the equation $\Phi_X(\theta) = \alpha$, for $\alpha \ge 0$. This was already used in various articles dealing with option pricing and risk management within the class of mixed-exponential jump-diffusion models (cf. among others [Ca09], [CCW09], [CK11], [CK12]). In this context, the following (important) lemma was derived in [Ca09] under hyper-exponential jump-diffusion models. The interested reader is referred for a proof to the latter article.

Lemma 5.2. *Let $\sigma_X > 0$ and $\Phi_X(\cdot)$ be defined as in (5.3.7). Then, for any $\alpha > 0$, the equation $\Phi_X(\theta) = \alpha$ has $(m + n + 2)$ real roots $\beta_{1,\alpha}, \dots, \beta_{m+1,\alpha}$ and $\gamma_{1,\alpha}, \dots, \gamma_{n+1,\alpha}$ that satisfy*

$$-\infty < \gamma_{n+1,\alpha} < -\eta_n < \gamma_{n,\alpha} < -\eta_{n-1} < \cdots < \gamma_{2,\alpha} < -\eta_1 < \gamma_{1,\alpha} < 0, \tag{5.3.8}$$

$$0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \cdots < \xi_{m-1} < \beta_{m,\alpha} < \xi_m < \beta_{m+1,\alpha} < \infty. \tag{5.3.9}$$

Remark 5.4.

i) At this point, one should note that the roots in Lemma 5.2 are only known in analytical form in very few cases. Nevertheless, this does not impact the importance and practicability of this result since all roots can be anyway recovered using standard numerical techniques.

ii) Similar characterizations to the one presented in Lemma 5.2 can be derived under the assumption that $\sigma_X = 0$ (cf. [FMV19]) and combining these characterizations with the upcoming derivations of Section 5.3.2 subsequently allows to derive semi-analytical pricing results under hyper-exponential jump-diffusion markets with $\sigma_X = 0$. However, since the main techniques do not substantially differ from the ones presented in this article, we refrain from discussing this type of results and focus on the more important case where $\sigma_X > 0$.[10]

♦

5.3.2 Maturity-Randomization and OIDEs

We now go back to the OIDE characterizations of Proposition 5.4, Proposition 5.5, and Proposition 5.6, and consider the respective problems (5.2.64)-(5.2.65), (5.2.75)-(5.2.76), and (5.2.86)-(5.2.91) for (regular) geometric down-and-out step call options under hyper-exponential jump-diffusion markets with $\sigma_X > 0$. First, we note that the infinitesimal generator (5.2.8) simplifies in this case to

$$\mathcal{A}_S V(\mathcal{T},x) = \frac{1}{2}\sigma_X^2 x^2 \partial_x^2 V(\mathcal{T},x) + (r-\delta-\lambda\zeta)x\partial_x V(\mathcal{T},x) + \lambda \int_{\mathbb{R}} \big(V(\mathcal{T},xe^y) - V(\mathcal{T},x)\big) f_{J_1}(y)dy. \quad (5.3.10)$$

Together with the properties of the hyper-exponential density $f_{J_1}(\cdot)$, this allows us to uniquely solve the problems (5.2.64)-(5.2.65), (5.2.75)-(5.2.76), and (5.2.86)-(5.2.91), and to derive closed-form expressions for the (regular) maturity-randomized geometric down-and-out step contracts $\widehat{\mathcal{DOSC}_E^{\star}}(\cdot)$, $\widehat{\mathcal{DOSC}_A^{\star}}(\cdot)$, and corresponding early exercise premiums $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{\star}}(\cdot)$, $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{0,\star}}(\cdot)$, and $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{\mathcal{J},\star}}(\cdot)$. This is discussed next.

We start by dealing with the maturity-randomized European-type contract $\widehat{\mathcal{DOSC}_E^{\star}}(\cdot)$. Here, upon imposing a natural smooth-fit condition (cf. among others [CCW10], [XY13], [WZ16]), the following characterization of the (regular) maturity-randomized European-type geometric down-and-out step call option $\widehat{\mathcal{DOSC}_E^{\star}}(\cdot)$ can be obtained. A proof is provided in Appendix B (cf. Section 5.6.2).

Proposition 5.7. *Consider a hyper-exponential jump-diffusion market as described by (5.2.1), (5.2.2), and (5.3.2), (5.3.3). Then, for any intensity parameter $\vartheta > 0$, the (regular) maturity-randomized European-type geometric down-and-out step call, $\widehat{\mathcal{DOSC}_E^{\star}}(\cdot)$, has the following representation*

$$\widehat{\mathcal{DOSC}_E^{\star}}(\vartheta,x;K,L,\rho_L) = \begin{cases} \sum\limits_{s=1}^{m+1} A_s^+ \left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta-\rho_L)}}, & 0 \le x < L, \\ \sum\limits_{s=1}^{m+1} B_s^+ \left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta)}} + \sum\limits_{u=1}^{n+1} B_u^- \left(\frac{x}{K}\right)^{\gamma_{u,(r+\vartheta)}}, & L \le x \le K, \\ \sum\limits_{u=1}^{n+1} C_u^- \left(\frac{x}{K}\right)^{\gamma_{u,(r+\vartheta)}} + \vartheta\left(\frac{x}{\delta+\vartheta} - \frac{K}{r+\vartheta}\right), & K < x < \infty, \end{cases} \quad (5.3.11)$$

where the vector of coefficients $\mathbf{v} := (A_1^+,\dots,A_{m+1}^+,B_1^+,\dots,B_{m+1}^+,B_1^-,\dots,B_{n+1}^-,C_1^-,\dots,C_{n+1}^-)^\intercal$ solves the system of equations given in (A.5.78) of Appendix B (cf. Section 5.6.2).

We next derive (semi-)analytical results for the (regular) maturity-randomized American-type geometric down-and-out step call contract $\widehat{\mathcal{DOSC}_A^{\star}}(\cdot)$. Having already obtained a closed-form expression for the

[10] A discussion of results for $\sigma_X = 0$ in a slightly different context is provided in [FMV19].

European-type option $\widehat{\mathcal{DOSC}}_E^\star(\cdot)$, we can now focus on the maturity-randomized early exercise pricing problem instead. Indeed, although a direct application of the techniques developed in the proof of Proposition 5.7 to $\widehat{\mathcal{DOSC}}_A^\star(\cdot)$ is equally feasible, switching to the maturity-randomized early exercise pricing problem substantially reduces the complexity of the resulting equations. We therefore follow this approach and decompose the American-type contract $\widehat{\mathcal{DOSC}}_A^\star(\cdot)$ as sum of the European-type option $\widehat{\mathcal{DOSC}}_E^\star(\cdot)$ and the early exercise premium $\widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\cdot)$. Additionally, since we have seen in Section 5.2 that the stopping region of a (maturity-randomized) geometric knock-out option is a sub-domain of the stopping region for the respective (maturity-randomized) barrier-type knock-out option, we can follow the ansatz in [XY13] (cf. [LV17], [CV18]) and conjecture that the early-exercise region is delimited by a free-boundary $\mathfrak{b}_s > K$, whose value has to be found. Combining these observations, we therefore arrive at the next proposition, whose proof is provided in Appendix B (cf. Section 5.6.2).

Proposition 5.8. *Consider a hyper-exponential jump-diffusion market as described by (5.2.1), (5.2.2), and (5.3.2), (5.3.3). Then, for any intensity parameter $\vartheta > 0$, the (regular) maturity-randomized American-type geometric down-and-out step call option, $\widehat{\mathcal{DOSC}}_A^\star(\cdot)$, is given by*

$$\widehat{\mathcal{DOSC}}_A^\star(\vartheta, x; K, L, \rho_L) = \widehat{\mathcal{DOSC}}_E^\star(\vartheta, x; K, L, \rho_L) + \widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\vartheta, x; K, L, \rho_L), \tag{5.3.12}$$

where the maturity-randomized early exercise premium to the (regular) geometric down-and-out step call, $\widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\cdot)$, has the following representation:

$$\widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\vartheta, x; K, L, \rho_L) = \begin{cases} \sum\limits_{s=1}^{m+1} D_s^+ \left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta-\rho_L)}}, & 0 \leq x < L, \\ \sum\limits_{s=1}^{m+1} F_s^+ \left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta)}} + \sum\limits_{u=1}^{n+1} F_u^- \left(\frac{x}{\mathfrak{b}_s}\right)^{\gamma_{u,(r+\vartheta)}}, & L \leq x < \mathfrak{b}_s, \\ x - K - \widehat{\mathcal{DOSC}}_E^\star(\vartheta, x; K, L, \rho_L), & \mathfrak{b}_s \leq x < \infty. \end{cases} \tag{5.3.13}$$

Here, the vector of coefficients $\mathbf{w} := (D_1^+, \ldots, D_{m+1}^+, F_1^+, \ldots, F_{m+1}^+, F_1^-, \ldots, F_{n+1}^-)^\intercal$ solves the system of equations given in (A.5.106) of Appendix B (cf. Section 5.6.2) and the early exercise boundary $\mathfrak{b}_s$ is implicitly given by combining (A.5.106) with Equation (A.5.119).

To complete our derivations, we lastly generalize the results obtained in [LV17] to American-type geometric step contracts and provide a jump-diffusion disentanglement of the maturity-randomized early exercise premium to the (regular) geometric down-an-out step call. Here, combining our results in Proposition 5.6 with ideas similarly employed in [LV17], [CV18], and [FMV19], allows us to derive (semi-)analytical expressions for $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{0,\star}}(\cdot)$ and $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{\mathcal{J},\star}}(\cdot)$, the maturity-randomized early exercise contribution of the diffusion and jump parts to the geometric down-and-out step call option. This leads to our final proposition, whose proof is provided in Appendix B (cf. Section 5.6.2).

Proposition 5.9. *Consider a hyper-exponential jump-diffusion market as described by (5.2.1), (5.2.2), and (5.3.2), (5.3.3). Then, for any intensity parameter $\vartheta > 0$, the maturity-randomized early exercise premium to the (regular) geometric down-and-out step call, $\widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\cdot)$, has the following decomposition*

$$\widehat{\mathcal{E}_{\mathcal{DOSC}}^\star}(\vartheta, x; K, L, \rho_L) = \widehat{\mathcal{E}_{\mathcal{DOSC}}^{0,\star}}(\vartheta, x; K, L, \rho_L) + \widehat{\mathcal{E}_{\mathcal{DOSC}}^{\mathcal{J},\star}}(\vartheta, x; K, L, \rho_L). \tag{5.3.14}$$

Here, the premiums $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{0,\star}}(\cdot)$ and $\widehat{\mathcal{E}_{\mathcal{DOSC}}^{\mathcal{J},\star}}(\cdot)$ refer to the maturity-randomized early exercise contributions of *the diffusion and jump parts, respectively, and are given by*

$$\widehat{\mathcal{E}^{0,\star}_{\mathcal{DOSC}}}(\vartheta,x;K,L,\rho_L) = \begin{cases} \sum_{s=1}^{m+1} D_s^{0,+}\left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta-\rho_L)}}, & 0 \leq x < L, \\ \sum_{s=1}^{m+1} F_s^{0,+}\left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta)}} + \sum_{u=1}^{n+1} F_u^{0-}\left(\frac{x}{\mathfrak{b}_s}\right)^{\gamma_{u,(r+\vartheta)}}, & L \leq x < \mathfrak{b}_s, \\ x - K - \widehat{\mathcal{DOSC}^{\star}_E}(\vartheta,x;K,L,\rho_L), & x = \mathfrak{b}_s, \\ 0, & \mathfrak{b}_s < x < \infty, \end{cases} \tag{5.3.15}$$

$$\widehat{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DOSC}}}(\vartheta,x;K,L,\rho_L) = \begin{cases} \sum_{s=1}^{m+1} D_s^{\mathcal{J},+}\left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta-\rho_L)}}, & 0 \leq x < L, \\ \sum_{s=1}^{m+1} F_s^{\mathcal{J},+}\left(\frac{x}{L}\right)^{\beta_{s,(r+\vartheta)}} + \sum_{u=1}^{n+1} F_u^{\mathcal{J},-}\left(\frac{x}{\mathfrak{b}_s}\right)^{\gamma_{u,(r+\vartheta)}}, & L \leq x < \mathfrak{b}_s, \\ 0, & x = \mathfrak{b}_s, \\ x - K - \widehat{\mathcal{DOSC}^{\star}_E}(\vartheta,x;K,L,\rho_L), & \mathfrak{b}_s < x < \infty, \end{cases} \tag{5.3.16}$$

where the two vectors of coefficients $\mathbf{w_0} := (D_1^{0,+},\ldots,D_{m+1}^{0,+},F_1^{0,+},\ldots,F_{m+1}^{0,+},F_1^{0,-},\ldots,F_{n+1}^{0,-})^\intercal$ *and* $\mathbf{w_J} := (D_1^{\mathcal{J},+},\ldots,D_{m+1}^{\mathcal{J},+},F_1^{\mathcal{J},+},\ldots,F_{m+1}^{\mathcal{J},+},F_1^{\mathcal{J},-},\ldots,F_{n+1}^{\mathcal{J},-})^\intercal$ *solve the systems of equations given in (A.5.124).*

5.4 Numerical Results

To complement the theoretical results of Section 5.2 and Section 5.3, we lastly illustrate structural and numerical properties of (regular) geometric down-and-out step call options under hyper-exponential jump-diffusion markets. For simplicity of the exposition as well as to allow for a better comparability of our results with the existing literature, we rely on Kou's double-exponential jump-diffusion model (cf. [Ko02]) as class representative and combine a variety of parameters that were similarly used in the following related articles: [Li99], [KW04], [CCW09], [CCW10], [CK12], [WZ16], [LV17], [CV18], and [DLM20]. All our numerical results are obtained using Matlab R2017b on an Intel CORE i7 processor.

5.4.1 Geometric Step Options and Limiting Contracts

We start our illustrations by investigating the convergence of geometric knock-out step call options to their limiting contracts. As already pointed out in Section 5.2, standard and (standard) barrier-type options can be understood as extremities on a continuum of geometric double barrier knock-out step contracts, namely when the knock-out rates are chosen as $\boldsymbol{\rho_\ell} = (0,0)$ and $\boldsymbol{\rho_\ell} = (-\infty,-\infty)$, respectively. Furthermore, since hyper-exponential jump-diffusion markets reduce to the Black & Scholes market (cf. [BS73]) when the jump intensity λ is zero, our results should be consistent in the limit $\lambda \downarrow 0$ with those obtained e.g. in [Li99] and [DLM20]. We verify these results in Table 5.1, where we compare the value of (regular) European-type and American-type geometric down-and-out step call options for $\rho_L = 0$ ("Standard Call Price"), $\rho_L = -26.34$ ("Step Call Price"), and $\rho_L = -50'000'000$ ("Barrier Call Price") with the respective Black & Scholes values ("B&S Values").[11] As in these papers (cf. also [WZ16]), we take $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $S_0 = 100$, $K = 100$, $L = 95$, and $\rho_L = -26.34$. Furthermore, we align the parameters of the double-exponential distribution to frequent choices in the literature and fix the probability of an up-jump with $p = 0.7$ (cf. [LV17]) and positive and negative jump parameters with $\xi = 25$ and $\eta = 50$, respectively (cf. [KW04],[CK12], [LV17], [CV18]). Finally, as in [WZ16] the convergence to the Black & Scholes values is investigated via $\lambda \in \{1, 0.1, 0.01, 0.001, 0.0001\}$.

[11] We compute the value of the American-type contracts under the Black & Scholes model using the algorithm in [DLM20] as well as Ritchken's trinomial tree method with $5'000$ time steps.

Table 5.1: Theoretical (down-and-out) call values and diffusion contributions to the early exercise premium for $r = 0.05$, $\delta = 0.07$, $S_0 = 100$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.7$, $\xi = 25$ and $\eta = 50$.

(Down-and-Out) Call Option Prices										
Parameters		*Standard Call Price*			*Step Call Price*			*Barrier Call Price*		
	λ	*Euro*	*Amer*	*DC (%)*	*Euro*	*Amer*	*DC (%)*	*Euro*	*Amer*	*DC (%)*
	1	6.833	7.040	91.52%	4.596	4.789	91.71%	3.374	3.551	91.88%
$S_0 = 100$	0.1	6.622	6.822	99.07%	4.519	4.706	99.09%	3.338	3.514	99.12%
$\sigma_X = 0.2$	0.01	6.600	6.800	99.91%	4.511	4.698	99.91%	3.334	3.510	99.91%
$\mathcal{T} = 1.0$	0.001	6.598	6.797	99.99%	4.510	4.697	99.99%	3.333	3.509	99.99%
	0.0001	6.598	6.797	100.00%	4.510	4.697	100.00%	3.333	3.509	100.00%
B&S Values	–	**6.598**	**6.885**	–	**4.511**	**4.745**	–	**3.332**	**3.529**	–
Rel. Error (%)	–	**0.001%**	**-1.277%**	–	**0.015%**	**-1.025%**	–	**0.025%**	**-0.568%**	–

As expected, the results in Table 5.1 show that standard options, geometric step options, and (standard) barrier-type options under the Black & Scholes market can be recovered by means of their respective contracts under double-exponential jump-diffusion markets as $\lambda \downarrow 0$. Furthermore, our results confirm the convergence of geometric down-and-out step call options to barrier-type down-and-out call contracts as $\rho_L \downarrow -\infty$. This becomes evident when looking at the "Barrier Call Price" of Table 5.1 while recalling that the Black & Scholes value is a true barrier-type value that was obtained using Ritchken's trinomial tree method and that the converging values correspond to those of geometric down-and-out step call options with $\rho_L = -50'000'000$. Finally, we note that our results are in line with the observations in [LV17], where the pricing accuracy of the Gaver-Stehfest inversion algorithm for European-type options was very high[12] and the relative pricing errors of the same inversion method applied to American-type options instead ranged from roughly $\pm 0.33\%$ to $\pm 1.38\%$. As explained in Remark 5.3., this is mainly due to the fact that maturity-randomized American-type contracts as well as maturity-randomized early exercise premiums do not anymore coincide with a strict application of the Laplace-Carson transform but are regardless treated as such.

5.4.2 Early Exercise Structure of Geometric Step Options with Jumps

Having verified the convergence of geometric step options to their limiting contracts, we next investigate the early exercise structure of (regular) geometric down-and-out step call options. To this end, we start by computing absolute European-type values ("Euro"), absolute early exercise premiums ("EEP"), relative early exercise contributions[13] ("EEP%"), and diffusion contributions to the early exercise premium ("DC%") for standard call options ("Standard Call Price"), (regular) geometric down-and-out step call options ("Step Call Price") and (regular) pseudo barrier-type down-and-out call options ("Barrier Call Price").[14] Here, we combine again the parameter choices in [Li99] and [DLM20] with frequent jump specifications in the literature. More specifically, we choose $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $S_0 \in \{90, 95, 100, 105, 110, 115\}$, $K = 100$, $L = 95$, $\rho_L = -26.24$ and fix the intensity measure Π_X in (5.3.5) by taking $\lambda \in \{5, 10\}$ (cf. [LV17], [CV18]), $p = 0.5$ (cf. [CCW09], [CCW10], [WZ16], [CV18]), and $(\xi, \eta) \in \{(50, 25), (50, 50), (25, 50), (25, 25)\}$ (cf. [KW04], [CK12], [LV17],[CV18]). The results are presented in Tables 5.2-5.5.

[12] In this article, the relative pricing errors of the Gaver-Stehfest inversion algorithm for European-type contracts never exceeded $\pm 0.22\%$.

[13] The relative early exercise contribution is expressed as percentage of the American-type geometric step option price.

[14] As earlier, we rely on results for geometric down-and-out step call contracts with $\rho_L = -50'000'000$ to derive pseudo barrier-type down-and-out call option values.

Table 5.2: Theoretical (down-and-out) call values and structure of the early exercise premium for $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 50$ and $\eta = 25$.

(Down-and-Out) Call Option Prices													
Parameters		*Standard Call Price*				*Step Call Price*				*Barrier Call Price*			
	S_0	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*
	90	3.500	0.062	1.74%	94.20%	0.268	0.009	3.07%	94.32%	0	0	–	–
(1)	95	5.241	0.112	2.09%	94.27%	1.757	0.059	3.23%	94.33%	0	0	–	–
$\sigma_X = 0.2$	100	7.416	0.190	2.50%	94.34%	4.992	0.178	3.45%	94.36%	3.686	0.165	4.28%	94.37%
$\lambda = 5.0$	105	10.011	0.305	2.96%	94.40%	8.309	0.330	3.82%	94.39%	7.305	0.353	4.61%	94.40%
$\mathcal{T} = 1.0$	110	12.992	0.469	3.48%	94.46%	11.804	0.535	4.34%	94.44%	11.037	0.597	5.13%	94.44%
	115	16.314	0.691	4.07%	94.52%	15.492	0.811	4.98%	94.50%	14.914	0.920	5.81%	94.54%
	90	4.098	0.065	1.57%	89.68%	0.344	0.010	2.79%	89.87%	0	0	–	–
(2)	95	5.933	0.113	1.87%	89.80%	2.012	0.061	2.93%	89.89%	0	0	–	–
$\sigma_X = 0.2$	100	8.169	0.186	2.22%	89.90%	5.413	0.175	3.12%	89.93%	3.990	0.161	3.88%	89.95%
$\lambda = 10.0$	105	10.791	0.290	2.62%	90.00%	8.791	0.314	3.44%	89.99%	7.683	0.334	4.17%	89.99%
$\mathcal{T} = 1.0$	110	13.767	0.435	3.06%	90.08%	12.313	0.497	3.88%	90.05%	11.442	0.552	4.60%	90.05%
	115	17.056	0.628	3.55%	90.17%	16.004	0.738	4.41%	90.12%	15.325	0.835	5.16%	90.13%

Table 5.3: Theoretical (down-and-out) call values and structure of the early exercise premium for $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 50$ and $\eta = 50$.

(Down-and-Out) Call Option Prices													
Parameters		*Standard Call Price*				*Step Call Price*				*Barrier Call Price*			
	S_0	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*
	90	3.163	0.064	1.98%	93.97%	0.232	0.008	3.46%	94.10%	0	0	–	–
(1)	95	4.835	0.117	2.37%	94.05%	1.588	0.060	3.63%	94.12%	0	0	–	–
$\sigma_X = 0.2$	100	6.958	0.202	2.82%	94.12%	4.679	0.188	3.87%	94.14%	3.432	0.174	4.82%	94.15%
$\lambda = 5.0$	105	9.523	0.328	3.33%	94.19%	7.949	0.355	4.28%	94.18%	6.983	0.382	5.18%	94.18%
$\mathcal{T} = 1.0$	110	12.498	0.509	3.91%	94.25%	11.430	0.583	4.85%	94.23%	10.702	0.654	5.76%	94.24%
	115	15.835	0.758	4.57%	94.33%	15.122	0.891	5.57%	94.31%	14.586	1.017	6.52%	94.43%
	90	3.441	0.068	1.94%	88.95%	0.268	0.009	3.40%	89.16%	0	0	–	–
(2)	95	5.155	0.121	2.30%	89.08%	1.685	0.062	3.55%	89.19%	0	0	–	–
$\sigma_X = 0.2$	100	7.303	0.204	2.72%	89.20%	4.836	0.190	3.77%	89.23%	3.522	0.174	4.71%	89.25%
$\lambda = 10.0$	105	9.875	0.325	3.19%	89.30%	8.138	0.352	4.14%	89.29%	7.107	0.377	5.04%	89.29%
$\mathcal{T} = 1.0$	110	12.839	0.497	3.72%	89.40%	11.636	0.569	4.66%	89.36%	10.845	0.638	5.56%	89.37%
	115	16.152	0.729	4.32%	89.53%	15.330	0.859	5.31%	89.46%	14.737	0.981	6.24%	89.57%

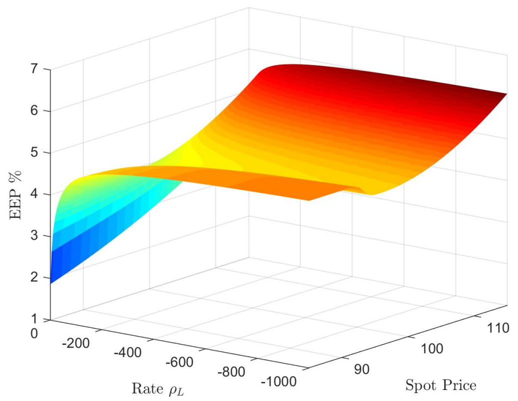
The results in Tables 5.2-5.5 show that the early exercise premium comprises a substantial part of the price of American-type geometric step contracts even if the option is out of the money. Additionally, they suggest that the absolute early exercise premium is for any rate ρ_L increasing in the underlying price S_0 and that the relative early exercise contribution tends to increase with more severe (i.e. more negative) knock-out rates. This is intuitively clear, since increasing the magnitude of the knock-out rate widens the early exercise domain of the American-type geometric step option and therefore further incentivizes early stopping. This subsequently raises the importance of the early exercise premium in the American-type geometric step option value and consequently increases its relative contribution. Next, we note that the diffusion contribution to the early exercise premium is a non-decreasing function of the underlying price S_0 and that this similarly seems to hold for the relative early exercise contribution. However, this last suggestion is wrong as can be seen in Figure 5.1a where we have plotted the relative early exercise contribution of the geometric down-and-out step call as a function of the underlying price $S_0 \in [85, 115]$ and the knock-out rate $\rho_L \in [-1000, 0]$ using the following standard parameters: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\lambda = 5$, $p = 0.5$, $\xi = 25$, $\eta = 50$. As it turns out, the general behavior of the relative early exercise contribution depends on the location of the spot price relative to the barrier level L. In particular, while the relative early exercise contribution is increasing in the underlying price S_0 above the barrier $L = 95$, it may be decreasing below the barrier for severe (i.e. large negative) knock-out rates ρ_L. Nevertheless, we note that the results in Figure 5.1 also confirm many of the properties already discussed. In particular, the monotonicity of the relative early exercise premium as function of the knock-out rate is clearly documented here. Additionally, Figure 5.1b provides further evidence for the monotonicity of the diffusion contribution to the early exercise premium as function of the underlying price S_0, while Figure 5.1c confirms the monotonicity of the absolute early exercise premium as function of the underlying price S_0.

Table 5.4: Theoretical (down-and-out) call values and structure of the early exercise premium for $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 25$ and $\eta = 50$.

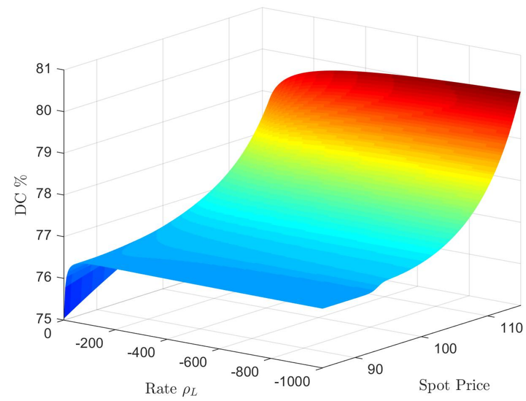
(Down-and-Out) Call Option Prices														
Parameters		*Standard Call Price*				*Step Call Price*				*Barrier Call Price*				
	S_0	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*
	90	3.645	0.080	2.15%	75.53%	0.294	0.012	3.75%	76.36%	0	0	–	–
(1)	95	5.362	0.137	2.49%	75.97%	1.685	0.067	3.82%	76.45%	0	0	–	–
$\sigma_X = 0.2$	100	7.501	0.222	2.88%	76.37%	4.854	0.202	3.99%	76.61%	3.506	0.182	4.94%	76.81%
$\lambda = 5.0$	105	10.054	0.345	3.31%	76.71%	8.177	0.368	4.31%	76.94%	7.110	0.391	5.21%	77.25%
$\mathcal{T} = 1.0$	110	12.994	0.514	3.80%	76.98%	11.685	0.585	4.77%	77.55%	10.861	0.652	5.67%	78.24%
	115	16.279	0.740	4.35%	77.14%	15.381	0.870	5.35%	78.79%	14.759	0.989	6.28%	80.55%
	90	4.347	0.096	2.16%	62.58%	0.391	0.015	3.78%	63.44%	0	0	–	–
(2)	95	6.141	0.155	2.45%	63.00%	1.865	0.074	3.82%	63.47%	0	0	–	–
$\sigma_X = 0.2$	100	8.321	0.238	2.78%	63.38%	5.152	0.212	3.94%	63.56%	3.649	0.188	4.89%	63.68%
$\lambda = 10.0$	105	10.878	0.354	3.15%	63.72%	8.549	0.374	4.19%	63.76%	7.328	0.393	5.09%	63.90%
$\mathcal{T} = 1.0$	110	13.788	0.508	3.55%	64.01%	12.099	0.577	4.55%	64.10%	11.126	0.640	5.44%	64.46%
	115	17.019	0.709	4.00%	64.18%	15.809	0.834	5.01%	64.74%	15.047	0.946	5.91%	65.93%

Table 5.5: Theoretical (down-and-out) call values and structure of the early exercise premium for $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 25$ and $\eta = 25$.

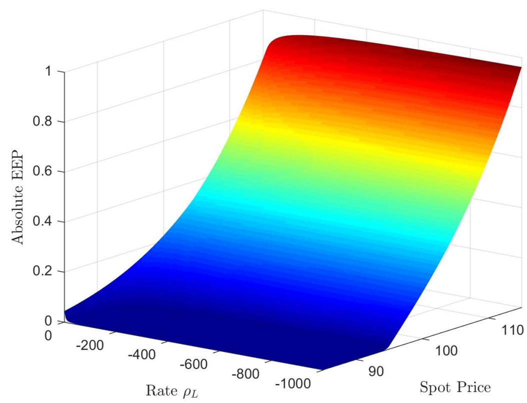
(Down-and-Out) Call Option Prices													
Parameters		*Standard Call Price*				*Step Call Price*				*Barrier Call Price*			
	S_0	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*	*Euro*	*EEP*	*EEP (%)*	*DC (%)*
	90	3.966	0.077	1.91%	76.61%	0.330	0.012	3.37%	77.40%	0	0	–	–
(1)	95	5.745	0.131	2.23%	77.04%	1.845	0.066	3.44%	77.48%	0	0	–	–
$\sigma_X = 0.2$	100	7.931	0.210	2.58%	77.42%	5.151	0.192	3.60%	77.62%	3.748	0.174	4.44%	77.79%
$\lambda = 5.0$	105	10.514	0.323	2.98%	77.75%	8.516	0.346	3.90%	77.89%	7.415	0.366	4.70%	78.11%
$\mathcal{T} = 1.0$	110	13.463	0.479	3.43%	78.01%	12.037	0.544	4.32%	78.35%	11.178	0.603	5.12%	78.81%
	115	16.740	0.685	3.93%	78.17%	15.730	0.803	4.85%	79.21%	15.069	0.907	5.68%	80.34%
	90	4.950	0.091	1.81%	64.97%	0.468	0.016	3.20%	65.77%	0	0	–	–
(2)	95	6.842	0.144	2.07%	65.37%	2.166	0.073	3.25%	65.81%	0	0	–	–
$\sigma_X = 0.2$	100	9.098	0.220	2.36%	65.74%	5.678	0.198	3.37%	65.91%	4.077	0.177	4.16%	66.01%
$\lambda = 10.0$	105	11.704	0.322	2.68%	66.08%	9.138	0.341	3.60%	66.09%	7.852	0.358	4.35%	66.17%
$\mathcal{T} = 1.0$	110	14.634	0.458	3.03%	66.37%	12.709	0.518	3.92%	66.34%	11.668	0.571	4.67%	66.50%
	115	17.857	0.632	3.42%	66.60%	16.419	0.741	4.32%	66.73%	15.583	0.834	5.08%	67.20%



(a) Relative Early Exercise Contribution.



(b) Diffusion Contribution.



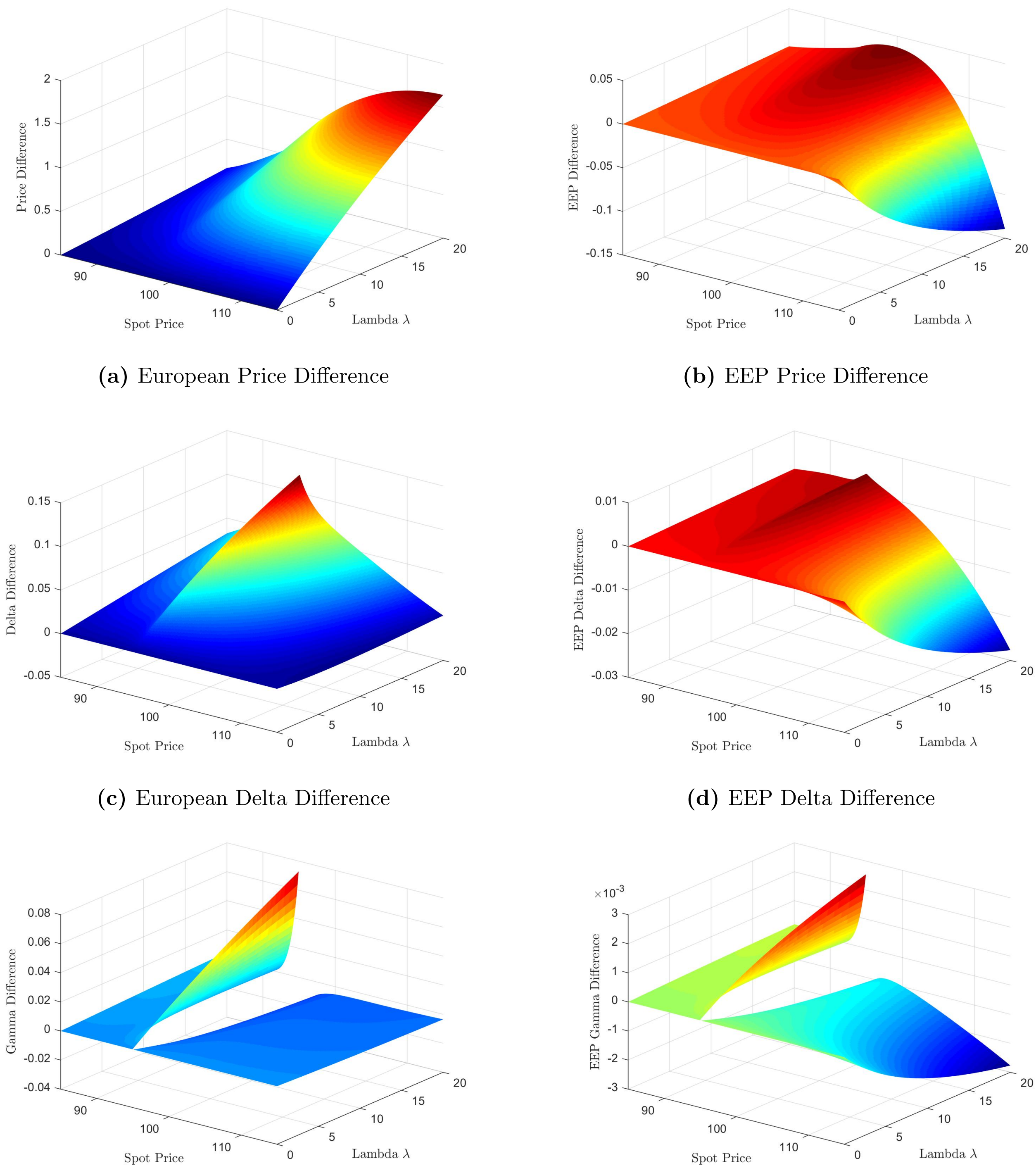
(c) Absolute Early Exercise Premium.

Figure 5.1: Relative early exercise contribution, diffusion contribution to the early exercise premium, and absolute early exercise premium of the geometric down-and-out step call as functions of the underlying price $S_0 \in [85, 115]$ and the knock-out rate $\rho_L \in [-1000, 0]$, when the remaining parameters are chosen as: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\lambda = 5$, $p = 0.5$, $\xi = 25$, $\eta = 50$.

5.4.3 The Impact of Jumps on Geometric Step Options

The vast majority of the geometric step option pricing literature either studies the Black & Scholes market (cf. [Li99], [DL02], [XY13], [DLM20]) or only European-type geometric step options under more advanced models (cf. [CCW10], [CMW13], [WZ16], [WZB17]). Additionally, although the inclusion of jumps naturally raises questions about their importance, no clear investigation of jump risk on the price and hedging parameters of geometric step options has been provided yet. This is the content of the next discussion.

We start by quantifying the impact of the jump intensity λ on the prices and greeks of (regular) geometric down-and-out step call options and of their respective early exercise premiums. Here, we plot in Figure 5.2 the difference in the prices, deltas, and gammas for the geometric down-and-out step call options with and without jumps as function of the underlying price $S_0 \in [85.115]$ and the intensity parameter $\lambda \in [0, 20]$ for the following parameters: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 25$, $\eta = 50$. As expected, all differences vanish as the jump parameter approaches zero and the value of



(a) European Price Difference

(b) EEP Price Difference

(c) European Delta Difference

(d) EEP Delta Difference

(e) European Gamma Difference

(f) EEP Gamma Difference

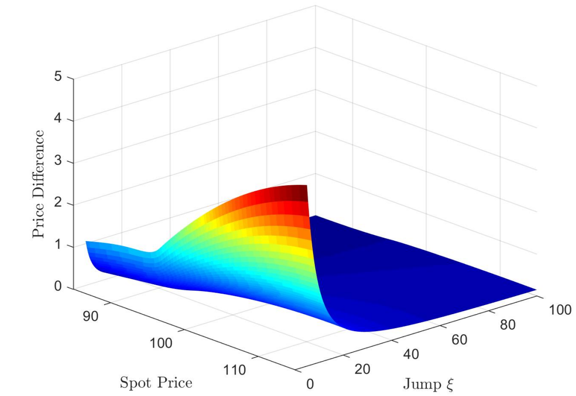
Figure 5.2: Difference in the prices, deltas, and gammas for the geometric down-and-out step calls with and without jumps as functions of the underlying price $S_0 \in [85.115]$ and the intensity parameter $\lambda \in [0, 20]$, when the remaining parameters are chosen as: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $p = 0.5$, $\xi = 25$, $\eta = 50$.

the European-type contracts increases when jumps are added (cf. Figure 5.2a). However, including jumps to the asset dynamics does not necessarily increase the value of the early exercise premium. This becomes evident when looking at Figure 5.2b where the difference in the early exercise premiums of the geometric down-and-out step calls with and without jumps becomes negative for out of the money options. Accordingly, the difference in the deltas of the European-type geometric step options with and without jumps is always positive (cf. Figure 5.2c) while the difference of the deltas for the corresponding early exercise premiums may become negative (cf. Figure 5.2d). Finally, one should note that the difference in the deltas attains for both European-type options and early exercise premiums its maximum at the barrier level L. These findings similarly hold true for the gamma differences, where the main (positive and negative) differences are found near the barrier (cf. Figure 5.2e and Figure 5.2f).

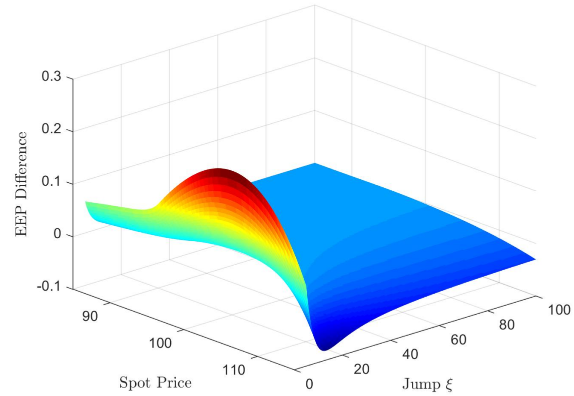
Secondly, we investigate the effect of the positive jump size ξ on the prices and greeks of (regular) geometric down-and-out step call options and of their respective early exercise premiums. This is demonstrated in Figure 5.3 where we have plotted the difference in the prices, deltas, and gammas for the geometric down-and-out step call options with and without jumps as functions of the underlying price $S_0 \in [85.115]$ and the positive jump parameter $\xi \in [5, 100]$ for the following specification: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $\lambda = 5$, $p = 0.5$, $\eta = 50$. Here, for a given spot S_0 the difference in prices of the geometric down-and-out step calls with and without jumps increases with increasing average jump size $\frac{1}{\xi}$ (cf. Figure 5.3a) and the same holds true for the difference in the early exercise premiums (cf. Figure 5.3b), except in parts of the payoff exercise domain, where an opposite relation is observed. While this result may seem surprising at first, it was already noticed for American-type Parisian options in [CV18], where the authors argue that the behavior is due to the structure of the early exercise premium, as difference between the intrinsic value of the option (which does not depend on the model parameters) and the corresponding European-type option price (which increases with increasing average jump size $\frac{1}{\xi}$). The same rationale also holds true in our case and the net effect then becomes negative in parts of the payoff exercise domain. Finally, an increase in the average jump size $\frac{1}{\xi}$ also usually leads to higher sensitivities for both European-type geometric down-and-out step calls and their respective early exercise premiums, except in parts of the payoff exercise domain where the same opposite relation is observed (cf. Figure 5.3c, Figure 5.3d, Figure 5.3e, and Figure 5.3f).

5.5 Conclusion

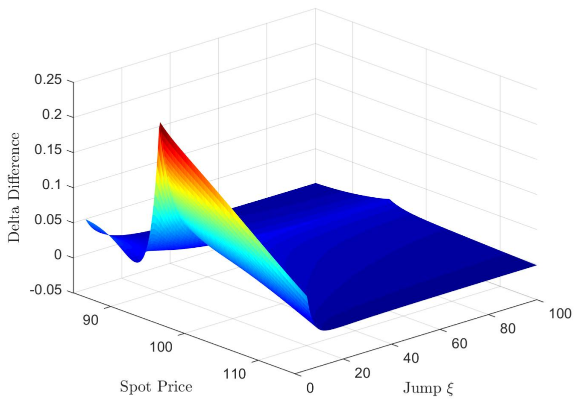
In the present article, we have extended the current literature on geometric step option pricing in several directions. Firstly, we have derived symmetry and parity relations and obtained various characterizations for both European-type and American-type geometric double barrier step options under exponential Lévy markets. In particular, we were able to translate the formalism introduced in [FMV19] to the setting of geometric double barrier step options and to generalize at the same time the ideas introduced in [CYY13], [LV17], [CV18] to Lévy-driven markets. As a result of these extensions, we were able to derive a jump-diffusion disentanglement for the early exercise premium of American-type geometric double barrier step options and its maturity-randomized equivalent as well as to characterize the diffusion and jump contributions to these early exercise premiums separately by means of partial integro-differential equations and ordinary integro-differential equations. To illustrate the practicability and importance of our characterizations, we have subsequently derived semi-analytical pricing results for (regular) European-type and American-type geometric down-and-out step call options under hyper-exponential jump-diffusion markets. Lastly, we have used the latter results to discuss the early exercise structure of geometric step options once jumps are added and to provide an analysis of the impact of jumps on the price and hedging parameters of (European-type and American-type) geometric step contracts.



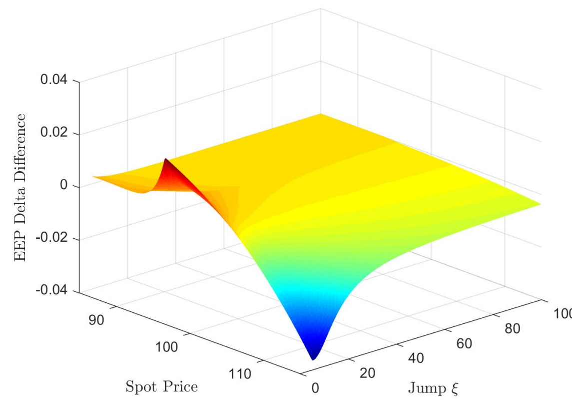
(a) European Price Difference



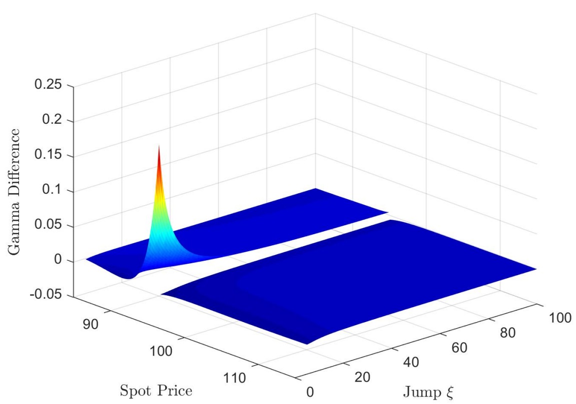
(b) EEP Price Difference



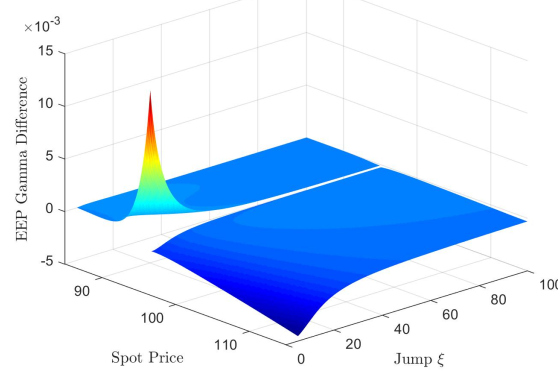
(c) European Delta Difference



(d) EEP Delta Difference



(e) European Gamma Difference



(f) EEP Gamma Difference

Figure 5.3: Difference in the prices, deltas, and gammas for the geometric down-and-out step calls with and without jumps as functions of the underlying price $S_0 \in [85.115]$ and the positive jump parameter $\xi \in [5, 100]$, when the remaining parameters are chosen as: $\mathcal{T} = 1.0$, $\sigma_X = 0.2$, $r = 0.05$, $\delta = 0.07$, $K = 100$, $L = 95$, $\rho_L = -26.34$, $\lambda = 5$, $p = 0.5$, $\eta = 50$.

5.6 Appendices

5.6.1 Appendix A: Proofs – Section 5.2

Proof of Lemma 5.1. For the sake of better exposition, we start by expanding our notation and define, for a Lévy process $(X_t)_{t\geq 0}$, $t \geq 0$, $x \geq 0$, $\gamma \geq 0$ and given barrier level $\ell > 0$,

$$\Gamma^{-}_{X,t,\ell}(x,\gamma) := \gamma + \int_0^t \mathbb{1}_{(0,\ell)}\big(xe^{X_s}\big)ds, \qquad \text{and} \qquad \Gamma^{+}_{X,t,\ell}(x,\gamma) := \gamma + \int_0^t \mathbb{1}_{(\ell,\infty)}\big(xe^{X_s}\big)ds. \tag{A.5.1}$$

Then, we denote by $(\tilde{X}_t)_{t\geq 0}$ the dual process to $(X_t)_{t\geq 0}$, i.e. the process defined for $t \geq 0$ by $\tilde{X}_t := -X_t$, and note that, for $t \geq 0$, $x \geq 0$, $K \geq 0$, $\gamma \geq 0$ and $\ell > 0$, the following relation holds

$$\Gamma^{\pm}_{X,t,\ell}(x,\gamma) = \Gamma^{\mp}_{\tilde{X},t,\frac{xK}{\ell}}(K,\gamma). \tag{A.5.2}$$

Combining (A.5.2) with the change of measure defined by the (1-)Esscher transform[15]

$$Z_t := \left.\frac{d\mathbb{Q}^{(1)}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} := \frac{e^{1\cdot X_t}}{\mathbb{E}^{\mathbb{Q}}\left[e^{1\cdot X_t}\right]} = e^{X_t - t\Phi_X(1)}, \tag{A.5.3}$$

allows us to recover (with $\delta = r - \Phi_X(1)$) that for any $T > 0$ and stopping time $\tau \in \mathfrak{T}_{[0,T]}$

$$\begin{aligned}
&\mathcal{DSC}\big(\tau, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\Big[B_\tau(r)^{-1}\exp\big\{\rho_L\Gamma^{-}_{X,\tau,L}(x,\gamma_L) + \rho_H\Gamma^{+}_{X,\tau,H}(x,\gamma_H)\big\}\big(xe^{X_\tau} - K\big)^{+}\Big] \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\Big[Z_\tau B_\tau(\delta)^{-1}\exp\big\{\rho_H\Gamma^{-}_{\tilde{X},\tau,\frac{xK}{H}}(K,\gamma_H) + \rho_L\Gamma^{+}_{\tilde{X},\tau,\frac{xK}{L}}(K,\gamma_L)\big\}\big(x - Ke^{\tilde{X}_\tau}\big)^{+}\Big] \\
&\qquad = \mathbb{E}^{\mathbb{Q}^{(1)}}\Big[B_\tau(\delta)^{-1}\exp\big\{\rho_H\Gamma^{-}_{\tilde{X},\tau,\frac{xK}{H}}(K,\gamma_H) + \rho_L\Gamma^{+}_{\tilde{X},\tau,\frac{xK}{L}}(K,\gamma_L)\big\}\big(x - Ke^{\tilde{X}_\tau}\big)^{+}\Big]
\end{aligned} \tag{A.5.4}$$

holds. Therefore, if one shows that $(\tilde{X}_t)_{t\geq 0}$ is again a Lévy process under the measure $\mathbb{Q}^{(1)}$, (A.5.4) implies that

$$\mathcal{DSC}\big(\tau, x, \gamma_L, \gamma_H;\, r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) = \mathcal{DSP}\Big(\tau, K, \gamma_H, \gamma_L; \delta, r, x, \frac{xK}{H}, \frac{xK}{L}, \rho_H, \rho_L, \Psi^{(1)}_{\tilde{X}}(\cdot)\Big), \tag{A.5.5}$$

where $\Psi^{(1)}_{\tilde{X}}(\cdot)$ denotes the Lévy exponent of $(\tilde{X}_t)_{t\geq 0}$ under the measure $\mathbb{Q}^{(1)}$. In fact, showing that $(\tilde{X}_t)_{t\geq 0}$ is a Lévy process is not hard and can be done as in [Ma20] (see also [FM06]). To conclude, we therefore need to verify that $\Psi^{(1)}_{\tilde{X}} \equiv \Psi_Y$ holds, where $\Psi_Y(\cdot)$ satisfies (5.2.17) and is given as in (5.2.18). To this end, we first note that

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{i\theta\tilde{X}_1}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_1 e^{-i\theta X_1}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{i(-(\theta+i))X_1}\right]e^{-\Phi_X(1)} = e^{-(\Psi_X(-(\theta+i)) + \Phi_X(1))}. \tag{A.5.6}$$

[15] The Esscher transform was first introduced 1932 by Esscher and later established in the theory of option pricing by Gerber and Shiu (cf. [GS94]). For an economical interpretation of this pricing technique in the continuous-time framework, we refer to [GS94].

Therefore, the Lévy exponent of $(\tilde{X}_t)_{t\geq 0}$ under $\mathbb{Q}^{(1)}$ can be recovered as

$$\begin{aligned}
\Psi_{\tilde{X}}^{(1)}(\theta) &= \Psi_X\left(-(\theta+i)\right) + \Phi_X(1) \\
&= i\big(b_X + \sigma_X^2\big)\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} \big(e^y - e^{-i(\theta+i)y} - i\theta y \mathbb{1}_{\{|y|\leq 1\}}\big)\Pi_X(dy) \\
&= i\Big(b_X + \sigma_X^2 - \int_{\mathbb{R}} \big(1-e^y\big) y \mathbb{1}_{\{|y|\leq 1\}}\Pi_X(dy)\Big)\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} e^y\big(1 - e^{i\theta(-y)} + i\theta(-y)\mathbb{1}_{\{|y|\leq 1\}}\big)\Pi_X(dy) \\
&= i\Big(b_X + \sigma_X^2 - \int_{\mathbb{R}} \big(1-e^y\big) y \mathbb{1}_{\{|y|\leq 1\}}\Pi_X(dy)\Big)\theta + \frac{1}{2}\sigma_X^2\theta^2 + \int_{\mathbb{R}} \big(1 - e^{i\theta y} + i\theta y\mathbb{1}_{\{|y|\leq 1\}}\big)\Pi^\star(dy),
\end{aligned} \tag{A.5.7}$$

where $\Pi^\star(dy) := e^{-y}\,\Pi_{\tilde{X}}(dy)$ and the jump measure of the dual process $(\tilde{X}_t)_{t\geq 0}$ satisfies $\Pi_{\tilde{X}}(dy) = \Pi_X(-dy)$. Lastly, we can combine these results with Equation (5.2.7) to obtain that

$$b_Y = -\Big(b_X + \sigma_X^2 - \int_{\mathbb{R}} \big(1-e^y\big) y \mathbb{1}_{\{|y|\leq 1\}}\Pi_X(dy)\Big) = \delta - r - \frac{1}{2}\sigma_X^2 + \int_{\mathbb{R}} \big(1 - e^y + y\mathbb{1}_{\{|y|\leq 1\}}\big)\Pi^\star(dy). \tag{A.5.8}$$

This finalizes the proof. $\square$

Proof of Corollary 5.1. First, we note that Equation (5.2.22) is a direct consequence of Lemma 5.1, since taking $\tau \equiv \mathcal{T}$ in (5.2.16) directly provides the result for the European-type options, while the corresponding equality for American-type options is recovered from (5.2.16) by taking the supremum over the set $\mathfrak{T}_{[0,\mathcal{T}]}$. Therefore, we proceed with the proof of the second identity.

For the proof of (5.2.23), we note as in the proof of Lemma 5.1 that, for a Lévy process $(X_t)_{t\geq 0}$, $t\geq 0$, $x \geq 0$, $\gamma \geq 0$ and given barrier level $\ell > 0$, the following identity holds

$$\Gamma^{\pm}_{X,t,\ell}(x,\gamma) = \Gamma^{\pm}_{X,t,\frac{\ell}{xK}}\Big(\frac{1}{K},\gamma\Big), \tag{A.5.9}$$

where we have used the notation introduced in (A.5.1). Then, combining the latter relation with Lemma 5.1 allows us to recover, for $T > 0$ and any stopping time $\tau \in \mathfrak{T}_{[0,T]}$, that

$$\begin{aligned}
&\mathcal{DSC}\big(\tau, x, \gamma_L, \gamma_H; r, \delta, K, L, H, \rho_L, \rho_H, \Psi_X(\cdot)\big) \\
&\qquad = xK\cdot \mathbb{E}^{\mathbb{Q}}\left[B_\tau(r)^{-1}\exp\Big\{\rho_L \Gamma^{-}_{X,\tau,\frac{L}{xK}}\Big(\frac{1}{K},\gamma_L\Big) + \rho_H \Gamma^{+}_{X,\tau,\frac{H}{xK}}\Big(\frac{1}{K},\gamma_H\Big)\Big\}\Big(\frac{1}{K}e^{X_\tau} - \frac{1}{x}\Big)^+\right] \\
&\qquad = xK\cdot \mathcal{DSC}\Big(\tau, \frac{1}{K}, \gamma_L, \gamma_H; r, \delta, \frac{1}{x}, \frac{L}{xK}, \frac{H}{xK}, \rho_L, \rho_H, \Psi_X(\cdot)\Big) \\
&\qquad = xK\cdot \mathcal{DSP}\Big(\tau, \frac{1}{x}, \gamma_H, \gamma_L; \delta, r, \frac{1}{K}, \frac{1}{H}, \frac{1}{L}, \rho_H, \rho_L, \Psi_Y(\cdot)\Big).
\end{aligned} \tag{A.5.10}$$

Here, $\Psi_Y(\cdot)$ represents, as in Lemma 5.1, the Lévy exponent of a process $(Y_t)_{t\geq 0}$ driving another exponential Lévy market and that satisfies the relations (5.2.17)-(5.2.21). Therefore, taking as earlier $\tau \equiv \mathcal{T}$ in (A.5.10) directly provides us with the result for the European-type options, while the corresponding identity for American-type contracts is obtained from (A.5.10) by taking the supremum over the set $\mathfrak{T}_{[0,\mathcal{T}]}$. $\square$

Proof of Proposition 5.1. We start by showing the continuity of $(\mathcal{T},x) \mapsto \mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ on the domain $[0,T]\times[0,\infty)$ for any $K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$. To do this, we first note that the continuity of the occupation times $x \mapsto \Gamma^{\pm}_{X,\mathcal{T},\ell}(x,0)$, defined for any $\mathcal{T}\in[0,T]$ and $\ell \geq 0$ as in (A.5.1), and the continuity of the function $x \mapsto (x-K)^+$, for $K\geq 0$, directly give by means of the dominated convergence theorem the continuity of $x \mapsto \mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ for any of the parameters $\mathcal{T},K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$. Therefore, to prove that $(\mathcal{T},x)\mapsto \mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ is, for any parameters $K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, continuous on $[0,T]\times[0,\infty)$, it is enough to show that $\mathcal{T}\mapsto \mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ is, for any parameters $x,K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, uniformly continuous on $[0,T]$. To obtain this property, we fix times to maturity $0\leq u<t\leq T$, recall that $\rho_L,\rho_H\leq 0$ and derive that

$$\begin{aligned}
&\left|\mathcal{DSC}^\star_E(t,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \mathcal{DSC}^\star_E(u,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})\right| \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}_x\left[e^{-ru+\rho_L\Gamma^-_{u,L}+\rho_H\Gamma^+_{u,H}} \left| e^{-\int_u^t\left(r-\rho_L\mathbb{1}_{(0,L)}(S_s)-\rho_H\mathbb{1}_{(H,\infty)}(S_s)\right)ds}(S_t-K)^+ - (S_u-K)^+\right|\right] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}_x\left[\left| e^{-\int_u^t\left(r-\rho_L\mathbb{1}_{(0,L)}(S_s)-\rho_H\mathbb{1}_{(H,\infty)}(S_s)\right)ds}(S_t-K) - (S_u-K)\right|\right] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}_x\left[S_u\left| S_tS_u^{-1} e^{-\int_u^t\left(r-\rho_L\mathbb{1}_{(0,L)}(S_s)-\rho_H\mathbb{1}_{(H,\infty)}(S_s)\right)ds} - 1\right|\right] + K\mathbb{E}^{\mathbb{Q}}_x\left[\left| e^{-\int_u^t\left(r-\rho_L\mathbb{1}_{(0,L)}(S_s)-\rho_H\mathbb{1}_{(H,\infty)}(S_s)\right)ds} - 1\right|\right] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}\left[xe^{X_u}\right]\left(\mathbb{E}^{\mathbb{Q}}\left[\left|e^{X_{t-u}+\lambda^\star(t-u)}-1\right|\right] + \mathbb{E}^{\mathbb{Q}}\left[\left|e^{X_{t-u}-\lambda^\star(t-u)}-1\right|\right]\right) + K\left(1-e^{-\lambda^\star(t-u)}\right) \\
&\quad \leq x\max\left\{1,\, e^{\Phi_X(1)T}\right\}\left(\mathbb{E}^{\mathbb{Q}}\left[\left|e^{X_{t-u}+\lambda^\star(t-u)}-1\right|\right] + \mathbb{E}^{\mathbb{Q}}\left[\left|e^{X_{t-u}-\lambda^\star(t-u)}-1\right|\right]\right) + K\left(1-e^{-\lambda^\star(t-u)}\right),
\end{aligned} \tag{A.5.11}$$

where $\lambda^\star := r-\rho_L-\rho_H$. Consequently, the right-continuity of the process $(X_t)_{t\in[0,T]}$ implies the convergence

$$\mathcal{DSC}^\star_E(t,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) - \mathcal{DSC}^\star_E(u,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) \to 0, \qquad \text{whenever } \; t-u\to 0. \tag{A.5.12}$$

This shows that the function $\mathcal{T}\mapsto \mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ is, for any parameters $x,K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, uniformly continuous over $[0,T]$ and the proof of the initial claim is complete.

We now prove that $\mathcal{DSC}^\star_E(\cdot)$ solves Equation (5.2.26) on $(0,T]\times[0,\infty)$ with initial condition (5.2.27). Here, we start by noting that, for any parameters $\mathcal{T},x,K,\boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, geometric double barrier step options can be rewritten in the simpler form

$$\mathcal{DSC}^\star_E(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \mathbb{E}^{\mathbb{Q}}_x\left[B_{\mathcal{T}}(r)^{-1}\, e^{\rho_L\Gamma^-_{\mathcal{T},L}+\rho_H\Gamma^+_{\mathcal{T},H}}\;(S_{\mathcal{T}}-K)^+\right] = \mathbb{E}^{\mathbb{Q}}_x\left[\left(\bar{S}_{\mathcal{T}}-K\right)^+\right], \tag{A.5.13}$$

where $(\bar{S}_t)_{t\in[0,T]}$ refers to the (strong) Markov process[16] obtained by "killing" the sample path of $(S_t)_{t\in[0,T]}$ at the proportional rate $\lambda(x) := r-\boldsymbol{\rho_\ell}\cdot\begin{pmatrix}\mathbb{1}_{(0,L)}(x)\\ \mathbb{1}_{(H,\infty)}(x)\end{pmatrix}$. The process' transition probabilities are then given by

$$\mathbb{Q}_x\left(\bar{S}_t\in A\right) = \mathbb{E}^{\mathbb{Q}}_x\left[e^{-\int_0^t\lambda(S_s)ds}\,\mathbb{1}_A(S_t)\right] \tag{A.5.14}$$

and we identify its cemetery state, without loss of generality, with $\partial\equiv 0$. Consequently, for any initial value $z=(\mathbf{t},x)\in[0,T]\times[0,\infty)$, the process $(Z_t)_{t\in[0,\mathbf{t}]}$ defined via $Z_t := (\mathbf{t}-t,\bar{S}_t)$, $\bar{S}_0 = x$, is a strong Markov

[16] It is well-known (cf. [PS06]) that the process $(\bar{S}_t)_{t\in[0,T]}$ defined this way preserves the (strong) Markov property of the underlying process $(S_t)_{t\in[0,T]}$.

process with state domain given by $\mathcal{D}_{\mathbf{t}} := [0,\mathbf{t}] \times [0,\infty)$. Additionally, $\mathcal{DSC}^{\star}_{E}(\cdot)$ can be re-expressed, for any $K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, as

$$\mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = V_E\big((\mathcal{T}, x)\big), \tag{A.5.15}$$

where the value function $V_E(\cdot)$ has the following representation under the measure $\mathbb{Q}^Z_z$ having initial distribution $Z_0 = z$:

$$V_E(z) := \mathbb{E}^{\mathbb{Q}^Z}_z\big[G(Z_{\tau_\mathcal{S}})\big], \qquad G(z) := (x-K)^+, \tag{A.5.16}$$

and $\tau_\mathcal{S} := \inf\{t \geq 0 : Z_t \in \mathcal{S}\}$, $\mathcal{S} := \big(\{0\} \times [0,\infty)\big) \cup \big([0,\mathbf{t}] \times \{0\}\big)$, is a stopping time that satisfies $\tau_\mathcal{S} \leq \mathbf{t}$, under $\mathbb{Q}^Z_z$ with $z = (\mathbf{t}, x)$. Furthermore, the stopping region $\mathcal{S}$ is for any $\mathbf{t} \in [0,T]$ a closed set in $\mathcal{D}_{\mathbf{t}}$. Therefore, standard arguments based on the strong Markov property of $(Z_t)_{t\in[0,\mathbf{t}]}$ (cf. [PS06]) imply that $V_E(\cdot)$ satisfies the following problem

$$\mathcal{A}_Z V_E(z) = 0, \qquad \text{on } \mathcal{D}_T \setminus \mathcal{S}, \tag{A.5.17}$$

$$V_E(z) = G(z), \qquad \text{on } \mathcal{S}, \tag{A.5.18}$$

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t\in[0,\mathbf{t}]}$. To complete the proof, we note that (for any suitable function $V : \mathcal{D}_{\mathbf{t}} \to \mathbb{R}$) the infinitesimal generator $\mathcal{A}_Z$ can be re-expressed as

$$\begin{aligned}\mathcal{A}_Z V\big((\mathbf{t},x)\big) &= -\partial_{\mathbf{t}} V\big((\mathbf{t},x)\big) + \mathcal{A}_{\bar{S}} V\big((\mathbf{t},x)\big) \\ &= -\partial_{\mathbf{t}} V\big((\mathbf{t},x)\big) + \mathcal{A}_S V\big((\mathbf{t},x)\big) - \lambda(x) V\big((\mathbf{t},x)\big).\end{aligned} \tag{A.5.19}$$

Therefore, recovering $\mathcal{DSC}^{\star}_{E}(\cdot)$ via (A.5.15) finally gives the required equation and initial condition. □

Proof of Proposition 5.2. First, we note that the continuity of $x \mapsto \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ for any $\mathcal{T}, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, follows, just like the continuity of $x \mapsto \mathcal{DSC}^{\star}_{E}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ for $\mathcal{T}, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, by means of the dominated convergence theorem while noticing the continuity of the occupation times $x \mapsto \Gamma^{\pm}_{X,\mathcal{T},\ell}(x,0)$, defined for any $\mathcal{T} \in [0,T]$ and $\ell \geq 0$ as in (A.5.1), and the continuity of the function $x \mapsto (x-K)^+$, for $K \geq 0$. Therefore, to prove that $(\mathcal{T}, x) \mapsto \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ is, for any parameters $K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, continuous on $[0,T] \times [0,\infty)$, it is enough to show that $\mathcal{T} \mapsto \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ is, for any parameters $x, K, \boldsymbol{\ell}$ and $\boldsymbol{\rho_\ell}$, uniformly continuous on $[0,T]$. To derive this property, we fix times to maturity $0 \leq u < t \leq T$, denote by τ_2 the optimal stopping time for $\mathcal{DSC}^{\star}_{A}(t, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ and set $\tau_1 := \tau_2 \wedge u$. Then, noting that $\mathcal{T} \mapsto \mathcal{DSC}^{\star}_{A}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ is a non-decreasing function[17] while recalling that $\rho_L, \rho_H \leq 0$ holds and that τ_1 is not necessarily optimal for the time to maturity u, we obtain that

$$\begin{aligned}
0 &\leq \mathcal{DSC}^{\star}_{A}(t, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \mathcal{DSC}^{\star}_{A}(u, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) \\
&\leq \mathbb{E}^{\mathbb{Q}}_x\left[e^{-r\tau_1 + \rho_L \Gamma^-_{\tau_1,L} + \rho_H \Gamma^+_{\tau_1,H}} \left(e^{-\int_{\tau_1}^{\tau_2}\left(r - \rho_L \mathbb{1}_{(0,L)}(S_s) - \rho_H \mathbb{1}_{(H,\infty)}(S_s)\right)ds} (S_{\tau_2} - K)^+ - (S_{\tau_1} - K)^+ \right) \right] \\
&\leq \mathbb{E}^{\mathbb{Q}}_x\left[\left| e^{-\int_{\tau_1}^{\tau_2}\left(r - \rho_L \mathbb{1}_{(0,L)}(S_s) - \rho_H \mathbb{1}_{(H,\infty)}(S_s)\right)ds} (S_{\tau_2} - K) - (S_{\tau_1} - K) \right| \right] \\
&\leq \mathbb{E}^{\mathbb{Q}}_x\left[S_{\tau_1} \left| S_{\tau_2} S_{\tau_1}^{-1} e^{-\int_{\tau_1}^{\tau_2}\left(r - \rho_L \mathbb{1}_{(0,L)}(S_s) - \rho_H \mathbb{1}_{(H,\infty)}(S_s)\right)ds} - 1 \right| \right] + K\big(1 - e^{-\lambda^\star (t-u)}\big) \\
&\leq x \max\big\{1,\, e^{\Phi_X(1)T}\big\} \left(\mathbb{E}^{\mathbb{Q}}\Big[\big| e^{X_{\tau_2 - \tau_1} + \lambda^\star (t-u)} - 1 \big| \Big] + \mathbb{E}^{\mathbb{Q}}\Big[\big| e^{X_{\tau_2 - \tau_1} - \lambda^\star (t-u)} - 1 \big| \Big] \right) + K\big(1 - e^{-\lambda^\star (t-u)}\big),
\end{aligned} \tag{A.5.20}$$

[17]This directly follows since, for $0 \leq \mathcal{T}_1 \leq \mathcal{T}_2 \leq T$, any stopping time $\tau \in \mathfrak{T}_{[0,\mathcal{T}_1]}$ also satisfies $\tau \in \mathfrak{T}_{[0,\mathcal{T}_2]}$.

where $\lambda^\star := r - \rho_L - \rho_H$. Therefore, since we have that $\tau_2 - \tau_1 \to 0$, for $t - u \to 0$, we obtain, by means of the dominated convergence theorem, the convergence

$$\mathcal{DSC}_A^\star(t, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) - \mathcal{DSC}_A^\star(u, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) \to 0, \qquad \text{whenever } \; t - u \to 0. \tag{A.5.21}$$

This finally shows that the function $\mathcal{T} \mapsto \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ is, for any parameters $x, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$, uniformly continuous over $[0, T]$ and the proof of the initial claim is complete.

To prove that $\mathcal{DSC}_A^\star(\cdot)$ satisfies the Cauchy-type problem (5.2.32), (5.2.33), we consider again, for any initial value $z = (\mathbf{t}, x) \in [0, T] \times [0, \infty)$, the (strong) Markov process $(Z_t)_{t \in [0, \mathbf{t}]}$ defined via $Z_t := (\mathbf{t} - t, \bar{S}_t)$, $\bar{S}_0 = x$, and make use of the fact that

$$\mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = V_A\big((\mathcal{T}, x)\big), \tag{A.5.22}$$

where $V_A(\cdot)$ is defined, under the measure $\mathbb{Q}_z^Z$ having initial distribution $Z_0 = z$, by

$$V_A(z) := \mathbb{E}_z^{\mathbb{Q}^Z}\big[G(Z_{\tau_{\mathcal{D}_s}})\big], \qquad G(z) := (x - K)^+, \tag{A.5.23}$$

and $\tau_{\mathcal{D}_s}$ refers to the optimal stopping time defined according to (5.2.31). Since $\tau_{\mathcal{D}_s} \le T$ and the stopping region $\mathcal{D}_s$ is a closed set in the domain $[0, T] \times [0, \infty)$,[18] this leads via standard arguments based on the strong Markov property of $(Z_t)_{t \in [0, \mathbf{t}]}$ (cf. [PS06]) to the following problem

$$\mathcal{A}_Z V_A(z) = 0, \qquad \text{on } \mathcal{D}_c, \tag{A.5.24}$$

$$V_A(z) = G(z), \qquad \text{on } \mathcal{D}_s, \tag{A.5.25}$$

and finally allows to recover the required equations (5.2.32) and (5.2.33) by means of Relations (A.5.22) and (A.5.19). □

Proof of Proposition 5.3. To start, we note that the strong Markov property of the process $(\bar{S}_t)_{t \in [0, T]}$ together with the optimality of the stopping time $\tau_{\mathcal{D}_s}$ defined, for any (fixed) $\mathcal{T} \in [0, T]$, according to (5.2.31) imply that the diffusion and jump contributions to the early exercise premium of the geometric double barrier step call, $\mathcal{E}_{\mathcal{DSC}}^{0,\star}(\cdot)$ and $\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}(\cdot)$ respectively, can be written in the form

$$\begin{aligned}\mathcal{E}_{\mathcal{DSC}}^{0,\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) &= \mathbb{E}_x^{\mathbb{Q}}\left[\Big(\big(\bar{S}_{\tau_{\mathcal{D}_s}} - K\big)^+ - \mathbb{E}_{\bar{S}_{\tau_{\mathcal{D}_s}}}^{\mathbb{Q}}\big[(\bar{S}_{\mathcal{T} - \tau_{\mathcal{D}_s}} - K)^+\big] \Big) \mathbb{1}_{\partial\mathcal{D}_s}\big((\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}})\big)\right] \\ &= \mathbb{E}_x^{\mathbb{Q}}\left[\Big(\big(\bar{S}_{\tau_{\mathcal{D}_s}} - K\big)^+ - \mathcal{DSC}_E^\star(\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}}; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) \Big) \mathbb{1}_{\partial\mathcal{D}_s}\big((\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}})\big)\right],\end{aligned} \tag{A.5.26}$$

$$\begin{aligned}\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) &= \mathbb{E}_x^{\mathbb{Q}}\left[\Big(\big(\bar{S}_{\tau_{\mathcal{D}_s}} - K\big)^+ - \mathbb{E}_{\bar{S}_{\tau_{\mathcal{D}_s}}}^{\mathbb{Q}}\big[(\bar{S}_{\mathcal{T} - \tau_{\mathcal{D}_s}} - K)^+\big] \Big) \mathbb{1}_{\mathcal{D}_s^\circ}\big((\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}})\big)\right] \\ &= \mathbb{E}_x^{\mathbb{Q}}\left[\Big(\big(\bar{S}_{\tau_{\mathcal{D}_s}} - K\big)^+ - \mathcal{DSC}_E^\star(\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}}; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) \Big) \mathbb{1}_{\mathcal{D}_s^\circ}\big((\mathcal{T} - \tau_{\mathcal{D}_s}, \bar{S}_{\tau_{\mathcal{D}_s}})\big)\right].\end{aligned} \tag{A.5.27}$$

Therefore, to prove that $\mathcal{E}_{\mathcal{DSC}}^{0,\star}(\cdot)$ and $\mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}(\cdot)$ satisfy Problem (5.2.50)-(5.2.52) and (5.2.53)-(5.2.55) respectively, we consider again, for any initial value $z = (\mathbf{t}, x) \in [0, T] \times [0, \infty)$, the (strong) Markov process $(Z_t)_{t \in [0, \mathbf{t}]}$ defined via $Z_t := (\mathbf{t} - t, \bar{S}_t)$, $\bar{S}_0 = x$, and make use of the fact that

$$\mathcal{E}_{\mathcal{DSC}}^{0,\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = V_{\mathcal{E}}^0\big((\mathcal{T}, x)\big), \quad \text{and} \quad \mathcal{E}_{\mathcal{DSC}}^{\mathcal{J},\star}(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = V_{\mathcal{E}}^{\mathcal{J}}\big((\mathcal{T}, x)\big), \tag{A.5.28}$$

[18] This directly follows from Representation (5.2.30) and the continuity of $(\mathcal{T}, x) \mapsto \mathcal{DSC}_A^\star(\mathcal{T}, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ on $[0, T] \times [0, \infty)$ for any $K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$.

where $V_{\mathcal{E}}^{0}(\cdot)$ and $V_{\mathcal{E}}^{\mathcal{J}}(\cdot)$ are defined, under the measure $\mathbb{Q}_z^Z$ having initial distribution $Z_0 = z = (\mathbf{t}, x)$, by

$$V_{\mathcal{E}}^{0}(z) := \mathbb{E}_z^{\mathbb{Q}^Z}\big[G_0(Z_{\tau_{\mathcal{D}_s}})\big], \qquad G_0\big((\mathbf{t},x)\big) := \big((x-K)^+ - \mathcal{DSC}_E^{\star}(\mathbf{t},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})\big)\,\mathbb{1}_{\partial\mathcal{D}_s}((\mathbf{t},x)), \tag{A.5.29}$$

$$V_{\mathcal{E}}^{\mathcal{J}}(z) := \mathbb{E}_z^{\mathbb{Q}^Z}\big[G_{\mathcal{J}}(Z_{\tau_{\mathcal{D}_s}})\big], \qquad G_{\mathcal{J}}\big((\mathbf{t},x)\big) := \big((x-K)^+ - \mathcal{DSC}_E^{\star}(\mathbf{t},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})\big)\,\mathbb{1}_{\mathcal{D}_s^{\circ}}((\mathbf{t},x)). \tag{A.5.30}$$

As earlier, since $\tau_{\mathcal{D}_s} \leq T$ and the stopping region $\mathcal{D}_s$ is a closed set in the domain $[0,T]\times[0,\infty)$, this leads via standard arguments based on the strong Markov property of $(Z_t)_{t\in[0,\mathbf{t}]}$ (cf. [PS06]) to the following problems

$$\mathcal{A}_Z V_{\mathcal{E}}^{0}(z) = 0, \qquad \text{on } \mathcal{D}_c, \tag{A.5.31}$$

$$V_{\mathcal{E}}^{0}(z) = G_0(z), \qquad \text{on } \mathcal{D}_s, \tag{A.5.32}$$

and

$$\mathcal{A}_Z V_{\mathcal{E}}^{\mathcal{J}}(z) = 0, \qquad \text{on } \mathcal{D}_c, \tag{A.5.33}$$

$$V_{\mathcal{E}}^{\mathcal{J}}(z) = G_{\mathcal{J}}(z), \qquad \text{on } \mathcal{D}_s, \tag{A.5.34}$$

and finally allows to recover the required equations (5.2.50)-(5.2.52) and (5.2.53)-(5.2.55) by means of Relations (A.5.28) and (A.5.19). □

Proof of Proposition 5.4. We start the proof of Proposition 5.4 by noting that the continuity of the maturity-randomized function $x \mapsto \widehat{\mathcal{DSC}_E^{\star}}(\vartheta,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ on $[0,\infty)$ directly follows from (5.2.62) and the continuity of $x \mapsto \mathcal{DSC}_E^{\star}(\mathcal{T},x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell})$ for $\mathcal{T}, K, \boldsymbol{\ell}$ and $\boldsymbol{\rho_\ell}$, by means of the dominated convergence theorem.[19] Therefore, we only need to establish that $\widehat{\mathcal{DSC}_E^{\star}}(\cdot)$ solves Equation (5.2.64) on $(0,\infty)$ with initial condition (5.2.65). To this end, we first recall that the (independent) exponentially distributed random time $\mathcal{T}_\vartheta$ can be viewed as the (first) jump time of a corresponding Poisson process $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$. Hence, for a fixed $\vartheta > 0$, we consider the process $(Z_t)_{t\geq 0}$ defined, for any initial value $z = (n,x) \in \mathbb{N}_0\times[0,\infty)$, via $Z_t := (n+N_t, \bar{S}_t)$, $\bar{S}_0 = x$, and note that it is a strong Markov process with state domain $\mathcal{D} := \mathbb{N}_0\times[0,\infty)$. Additionally, $\widehat{\mathcal{DSC}_E^{\star}}(\cdot)$ can be re-expressed, for $\vartheta, K, \boldsymbol{\ell}$ and $\boldsymbol{\rho_\ell}$, as

$$\widehat{\mathcal{DSC}_E^{\star}}(\vartheta,x;K,\boldsymbol{\ell},\boldsymbol{\rho_\ell}) = \widehat{V_E}\big((0,x)\big), \tag{A.5.35}$$

where the value function $\widehat{V_E}(\cdot)$ has the following representation under the measure $\mathbb{Q}_z^Z$ having initial distribution $Z_0 = z$:

$$\widehat{V_E}(z) := \mathbb{E}_z^{\mathbb{Q}^Z}\big[G(Z_{\tau_{\mathcal{S}}})\big], \qquad G(z) := (x-K)^+, \tag{A.5.36}$$

and $\tau_{\mathcal{S}} := \inf\{t\geq 0 : Z_t \in \mathcal{S}\}$, $\mathcal{S} := \big(\mathbb{N}\times[0,\infty)\big)\cup\big(\mathbb{N}_0\times\{0\}\big)$, is a $\mathbb{Q}_z^Z$-almost surely finite stopping time for any $z = (n,x)\in\mathcal{D}$.[20] Furthermore, the stopping region $\mathcal{S}$ forms (under an appropriate product-metric) a closed set in $\mathcal{D}$.[21] Therefore, standard arguments based on the strong Markov property of the process $(Z_t)_{t\geq 0}$ (cf. [PS06]) imply that $\widehat{V_E}(\cdot)$ satisfies the following problem

$$\mathcal{A}_Z \widehat{V_E}(z) = 0, \qquad \text{on } \mathcal{D}\setminus\mathcal{S}, \tag{A.5.37}$$

$$\widehat{V_E}(z) = G(z), \qquad \text{on } \mathcal{S}, \tag{A.5.38}$$

[19] Recall that we have assumed the integrability of the underlying price process $(S_t)_{t\geq 0}$.

[20] The finiteness of this stopping time directly follows from the finiteness of the first moment of any exponential distribution.

[21] We note that several choices of a product-metric on $\mathcal{D}$ give the closedness of the set $\mathcal{S}$. In particular, one may choose on $\mathbb{N}_0$ the following metric

$$d_{\mathbb{N}_0}(m,n) := \begin{cases} 1 + |2^{-m} - 2^{-n}|, & m \neq n, \\ 0, & m = n, \end{cases}$$

and consider the product-metric on $\mathcal{D}$ obtained by combining $d_{\mathbb{N}_0}(\cdot,\cdot)$ on $\mathbb{N}_0$ with the Euclidean metric on $[0,\infty)$.

where $\mathcal{A}_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t\geq 0}$. To complete the proof, we note that the infinitesimal generator $\mathcal{A}_Z$ can be re-expressed (for any suitable function $V : \mathcal{D} \to \mathbb{R}$) as

$$\begin{aligned}\mathcal{A}_Z V\big((n,x)\big) &= \mathcal{A}_N^n V\big((n,x)\big) + \mathcal{A}_{\bar{S}}^x V\big((n,x)\big) \\ &= \vartheta\left(V\big((n+1,x)\big) - V\big((n,x)\big)\right) + \mathcal{A}_S^x V\big((n,x)\big) - \lambda(x) V\big((n,x)\big),\end{aligned} \tag{A.5.39}$$

where $\mathcal{A}_N$ denotes the infinitesimal generator of the Poisson process $(N_t)_{t\geq 0}$ and the notation $\mathcal{A}_N^n$, $\mathcal{A}_{\bar{S}}^x$, and $\mathcal{A}_S^x$ is used to indicate that the generators are applied to n and x, respectively. Therefore, recovering $\widehat{\mathcal{DSC}_E^\star}(\cdot)$ via (A.5.35) while noting Relation (A.5.39) and the fact that for any $x \in [0,\infty)$ we have

$$\widehat{V_E}\big((1,x)\big) = G\big((1,x)\big) = (x-K)^+ \tag{A.5.40}$$

finally completes the proof. $\square$

Proof of Proposition 5.5. First, we note that the discussion preceding Proposition 5.5 implies that the optimal stopping problem (5.2.70) can be re-expressed, under the measure $\mathbb{Q}_z^Z$ having initial distribution $Z_0 = z \in \mathcal{D}$, as

$$\widehat{V_A}(z) = \mathbb{E}_z^{\mathbb{Q}^Z}\left[G\Big(Z^{\mathcal{S}_J}_{\tau_{\widehat{\mathcal{D}_s^{Gen.}}}}\Big)\right], \tag{A.5.41}$$

where $\tau_{\widehat{\mathcal{D}_s^{Gen.}}}$ is defined as in (5.2.72) and $G(z) := (x-K)^+$, for $z \in \mathcal{D}$. Additionally, the finiteness of the first moment of the exponential distribution for any $\vartheta > 0$ implies that this stopping time is $\mathbb{Q}_z^Z$-almost surely finite for any $z \in \mathcal{D}$, and combining this property with the closedness[22] of the stopping domain $\widehat{\mathcal{D}_s^{Gen.}}$ gives (cf. [PS06]) that $\widehat{V_A}(\cdot)$ satisfies the following problem

$$\mathcal{A}_Z \widehat{V_A}(z) = 0, \qquad \text{on } \widehat{\mathcal{D}_c^{Gen.}}, \tag{A.5.42}$$

$$\widehat{V_A}(z) = G(z), \qquad \text{on } \widehat{\mathcal{D}_s^{Gen.}}. \tag{A.5.43}$$

Consequently, recovering $\widehat{\mathcal{DSC}_A^\star}(\cdot)$ by means of Relation (5.2.69) while noting Identity (A.5.39) and the fact that

$$\widehat{\mathcal{D}_s^{Gen.}} = \mathcal{S}_J \cup \big(\{0\} \times \widehat{\mathcal{D}}_{\vartheta,s}\big) \tag{A.5.44}$$

and

$$\widehat{V_A}\big((1,x)\big) = G\big((1,x)\big) = (x-K)^+ \tag{A.5.45}$$

finally gives the required Equations (5.2.75) and (5.2.76).

The continuity of $x \mapsto \widehat{\mathcal{DSC}_A^\star}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell})$ on $[0,\infty)$ for $\vartheta, K, \boldsymbol{\ell}$, and $\boldsymbol{\rho_\ell}$ is an easy consequence of the continuity of $x \mapsto (x-K)^+$ and the dominated convergence theorem. This concludes the proof of the proposition. $\square$

Proof of Proposition 5.6. Following the ideas outlined in the previous proofs, we re-consider, for any $\vartheta > 0$ and initial value $z = (n,x) \in \mathbb{N}_0 \times [0,\infty)$, the process $(Z_t)_{t\geq 0}$ defined on the state domain $\mathcal{D} := \mathbb{N}_0 \times [0,\infty)$ via $Z_t := (n + N_t, \bar{S}_t)$, $\bar{S}_0 = x$, as well as its stopped version, $(Z_t^{\mathcal{S}_J})_{t\geq 0}$, defined according to (5.2.68) and note that the diffusion and jump contributions to the maturity-randomized early exercise

[22] As earlier, this property can be obtained under the product-metric considered in Footnote 21.

premium of the geometric double barrier step call, $\widehat{\mathcal{E}^{0,\star}_{\mathcal{DSC}}}(\cdot)$ and $\widehat{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}}(\cdot)$ respectively, can be re-expressed, using these processes, in the form

$$\widehat{\mathcal{E}^{0,\star}_{\mathcal{DSC}}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \widehat{V^0_{\mathcal{E}}}\big((0,x)\big), \quad \text{and} \quad \widehat{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DSC}}}(\vartheta, x; K, \boldsymbol{\ell}, \boldsymbol{\rho_\ell}) = \widehat{V^{\mathcal{J}}_{\mathcal{E}}}\big((0,x)\big), \tag{A.5.46}$$

where $\widehat{V^0_{\mathcal{E}}}(\cdot)$ and $\widehat{V^{\mathcal{J}}_{\mathcal{E}}}(\cdot)$ are defined, under the measure $\mathbb{Q}^Z_z$ having initial distribution $Z_0 = z$, by

$$\widehat{V^0_{\mathcal{E}}}(z) := \mathbb{E}^{\mathbb{Q}^Z}_z\left[\widehat{G_0}\Big(Z^{\mathcal{S}_J}_{\tau_{\widehat{\mathcal{D}^{Gen.}_s}}}\Big)\right], \quad \widehat{G_0}\big((n,x)\big) := \big((x-K)^+ - \widehat{V_E}\big((n,x)\big)\big)\, \mathbb{1}_{\partial\widehat{\mathcal{D}^{Gen.}_s}}\big((n,x)\big), \tag{A.5.47}$$

$$\widehat{V^{\mathcal{J}}_{\mathcal{E}}}(z) := \mathbb{E}^{\mathbb{Q}^Z}_z\left[\widehat{G_{\mathcal{J}}}\Big(Z^{\mathcal{S}_J}_{\tau_{\widehat{\mathcal{D}^{Gen.}_s}}}\Big)\right], \quad \widehat{G_{\mathcal{J}}}\big((n,x)\big) := \big((x-K)^+ - \widehat{V_E}\big((n,x)\big)\big)\, \mathbb{1}_{\big(\widehat{\mathcal{D}^{Gen.}_s}\big)^\circ}\big((n,x)\big). \tag{A.5.48}$$

As earlier, the $\mathbb{Q}^Z_z$-almost sure finiteness of the stopping time $\tau_{\widehat{\mathcal{D}^{Gen.}_s}}$ for any $z \in \mathcal{D}$ and the closedness[23] of the stopping domain $\widehat{\mathcal{D}^{Gen.}_s}$ lead via standard arguments (cf. [PS06]) to the following problems

$$\mathcal{A}_Z \widehat{V^0_{\mathcal{E}}}(z) = 0, \qquad \text{on } \widehat{\mathcal{D}^{Gen.}_c}, \tag{A.5.49}$$

$$\widehat{V^0_{\mathcal{E}}}(z) = \widehat{G_0}(z), \qquad \text{on } \widehat{\mathcal{D}^{Gen.}_s}, \tag{A.5.50}$$

and

$$\mathcal{A}_Z \widehat{V^{\mathcal{J}}_{\mathcal{E}}}(z) = 0, \qquad \text{on } \widehat{\mathcal{D}^{Gen.}_c}, \tag{A.5.51}$$

$$\widehat{V^{\mathcal{J}}_{\mathcal{E}}}(z) = \widehat{G_{\mathcal{J}}}(z), \qquad \text{on } \widehat{\mathcal{D}^{Gen.}_s}. \tag{A.5.52}$$

Finally, in view of (A.5.44), it is clear that[24]

$$\partial\widehat{\mathcal{D}^{Gen.}_s} = \mathcal{S}_J \cup \big(\{0\} \times \partial\widehat{\mathcal{D}}_{\vartheta,s}\big), \quad \text{and} \quad \big(\widehat{\mathcal{D}^{Gen.}_s}\big)^\circ = \mathcal{S}_J \cup \big(\{0\} \times \widehat{\mathcal{D}}^\circ_{\vartheta,s}\big), \tag{A.5.53}$$

so that

$$\widehat{V^0_{\mathcal{E}}}\big((1,x)\big) = \widehat{G_0}\big((1,x)\big) = 0 = \widehat{G_{\mathcal{J}}}\big((1,x)\big) = \widehat{V^{\mathcal{J}}_{\mathcal{E}}}\big((1,x)\big). \tag{A.5.54}$$

Therefore, combining these properties with Relations (A.5.46) and (A.5.39) finally allows to recover the required equations (5.2.86)-(5.2.88) and (5.2.89)-(5.2.91). This completes the proof. □

5.6.2 Appendix B: Proofs – Section 5.3

Proof of Proposition 5.7. For simplicity, we rewrite the price of the maturity-randomized European-type down-and-out step contract $\widehat{\mathcal{DOSC}^\star_E}(\cdot)$ as function of the log-price $\boldsymbol{x} := \log(x)$ and the log-strike $\boldsymbol{k} := \log(K)$ via $\overline{\mathcal{DOSC}^\star_E}(\cdot)$, i.e. we rely on the following relation

$$\overline{\mathcal{DOSC}^\star_E}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) := \widehat{\mathcal{DOSC}^\star_E}(\vartheta, e^{\boldsymbol{x}}; e^{\boldsymbol{k}}, L, \rho_L). \tag{A.5.55}$$

This transforms (5.2.64) into the following equation

$$\vartheta(e^{\boldsymbol{x}} - e^{\boldsymbol{k}})^+ + \mathcal{A}_X \overline{\mathcal{DOSC}^\star_E}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - \big(r + \vartheta - \rho_L \mathbb{1}_{(0,L)}(e^{\boldsymbol{x}})\big)\overline{\mathcal{DOSC}^\star_E}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \tag{A.5.56}$$

[23] e.g. under the product-metric considered in Footnote 21.
[24] e.g. under the product-metric considered in Footnote 21.

with $\mathcal{A}_X$ denoting the infinitesimal generator of $(X_t)_{t\geq 0}$, i.e.

$$\mathcal{A}_X V(\boldsymbol{x}) := \frac{1}{2}\sigma_X^2 \partial_{\boldsymbol{x}}^2 V(\boldsymbol{x}) + \Big(r - \delta - \lambda\zeta - \frac{1}{2}\sigma_X^2\Big)\partial_{\boldsymbol{x}} V(\boldsymbol{x}) + \lambda \int_{\mathbb{R}} \big(V(\boldsymbol{x}+\boldsymbol{y}) - V(\boldsymbol{x})\big) f_{J_1}(\boldsymbol{y}) d\boldsymbol{y}. \tag{A.5.57}$$

Equivalently, this can be written in the following system of three equations

$$\mathcal{A}_X \overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - (r + \vartheta - \rho_L)\overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \qquad \text{for } -\infty < \boldsymbol{x} < \ell^*, \tag{A.5.58}$$

$$\mathcal{A}_X \overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - (r + \vartheta)\overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \qquad \text{for } \ell^* \leq \boldsymbol{x} \leq \boldsymbol{k}, \tag{A.5.59}$$

$$\mathcal{A}_X \overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - (r + \vartheta)\overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = \vartheta(e^{\boldsymbol{k}} - e^{\boldsymbol{x}}), \qquad \text{for } \boldsymbol{k} < \boldsymbol{x} < \infty, \tag{A.5.60}$$

where we have set $\ell^* := \log(L)$. Combining the arguments provided in [CK11] (cf. also [LV17], [CV18], [FMV19]) with the fact that

$$P_1(x) := \vartheta\left(\frac{e^{\boldsymbol{x}}}{\delta + \vartheta} - \frac{e^{\boldsymbol{k}}}{r + \vartheta}\right)$$

is a particular solution to (A.5.60) implies that the general solution to (A.5.58)-(A.5.60) takes the following form

$$\overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = \begin{cases} \sum\limits_{s=1}^{m+1} A_s^+ e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(\boldsymbol{x}-\ell^*)}, & -\infty < \boldsymbol{x} < \ell^*, \\ \sum\limits_{s=1}^{m+1} B_s^+ e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{x}-\ell^*)} + \sum\limits_{u=1}^{n+1} B_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-\boldsymbol{k})}, & \ell^* \leq \boldsymbol{x} \leq \boldsymbol{k}, \\ \sum\limits_{u=1}^{n+1} C_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-\boldsymbol{k})} + \vartheta\left(\frac{e^{\boldsymbol{x}}}{\delta+\vartheta} - \frac{e^{\boldsymbol{k}}}{r+\vartheta}\right), & \boldsymbol{k} < \boldsymbol{x} < \infty, \end{cases} \tag{A.5.61}$$

where the coefficients $(A_s^+)_{s=1,\dots,m+1}$, $(B_s^+)_{s=1,\dots,m+1}$, $(B_u^-)_{u=1,\dots,n+1}$ and $(C_u^-)_{u=1,\dots,n+1}$ are subsequently determined by analyzing the solution under the respective equations and in the different regions. This is done next.

STEP 1: $-\infty < \boldsymbol{x} < \ell^*$.
To start we derive that

$$\begin{aligned} &\int_{\mathbb{R}} \overline{\mathcal{DOSC}}_E^{\star}(\vartheta, \boldsymbol{x} + y; \boldsymbol{k}, L, \rho_L) f_{J_1}(y) dy \\ &= \sum_{s=1}^{m+1}\sum_{j=1}^{n} q_j \eta_j e^{-\eta_j \boldsymbol{x}} A_s^+ \int_{-\infty}^{\boldsymbol{x}} e^{\eta_j z} e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(z-\ell^*)}\, dz + \sum_{s=1}^{m+1}\sum_{i=1}^{m} p_i \xi_i e^{\xi_i \boldsymbol{x}} A_s^+ \int_{\boldsymbol{x}}^{\ell^*} e^{-\xi_i z} e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(z-\ell^*)}\, dz \\ &+ \sum_{s=1}^{m+1}\sum_{i=1}^{m} p_i \xi_i e^{\xi_i \boldsymbol{x}} B_s^+ \int_{\ell^*}^{\boldsymbol{k}} e^{-\xi_i z} e^{\beta_{s,(r+\vartheta)}\cdot(z-\ell^*)}\, dz + \sum_{u=1}^{n+1}\sum_{i=1}^{m} p_i \xi_i e^{\xi_i \boldsymbol{x}} B_u^- \int_{\ell^*}^{\boldsymbol{k}} e^{-\xi_i z} e^{\gamma_{u,(r+\vartheta)}\cdot(z-\boldsymbol{k})}\, dz \\ &+ \sum_{u=1}^{n+1}\sum_{i=1}^{m} p_i \xi_i e^{\xi_i \boldsymbol{x}} C_u^- \int_{\boldsymbol{k}}^{\infty} e^{-\xi_i z} e^{\gamma_{u,(r+\vartheta)}\cdot(z-\boldsymbol{k})}\, dz + \sum_{i=1}^{m} p_i \xi_i e^{\xi_i \boldsymbol{x}} \left(\frac{\vartheta}{\delta+\vartheta}\int_{\boldsymbol{k}}^{\infty} e^{-(\xi_i - 1)\cdot z}\, dz - \frac{\vartheta e^{\boldsymbol{k}}}{r+\vartheta}\int_{\boldsymbol{k}}^{\infty} e^{-\xi_i z}\, dz\right). \end{aligned} \tag{A.5.62}$$

After some algebra, Equation (A.5.58) can be transformed to obtain

$$\sum_{s=1}^{m+1} A_s^+ e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(\boldsymbol{x}-\ell^*)} \underbrace{\left(\Phi_X\left(\beta_{s,(r+\vartheta-\rho_L)}\right)-(r+\vartheta-\rho_L)\right)}_{=0} + \lambda \sum_{i=1}^{m} p_i \xi_i e^{\xi_i(\boldsymbol{x}-\ell^*)}\, \mathcal{R}_i^1(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \tag{A.5.63}$$

where, for $i = 1,\ldots,m$,

$$\begin{aligned}\mathcal{R}_i^1(\vartheta;\boldsymbol{k},L,\rho_L) := -\sum_{s=1}^{m+1}&\left(A_s^+\frac{1}{\xi_i-\beta_{s,(r+\vartheta-\rho_L)}} - B_s^+\frac{1-e^{(\beta_{s,(r+\vartheta)}-\xi_i)(\boldsymbol{k}-\ell^*)}}{\xi_i-\beta_{s,(r+\vartheta)}}\right)\\ &+\sum_{u=1}^{n+1}\left(B_u^-\frac{e^{-\gamma_{u,(r+\vartheta)}(\boldsymbol{k}-\ell^*)}-e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i-\gamma_{u,(r+\vartheta)}} + C_u^-\frac{e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i-\gamma_{u,(r+\vartheta)}}\right)\\ &+\frac{\vartheta e^{\boldsymbol{k}}e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{(\xi_i-1)(\delta+\vartheta)} - \frac{\vartheta e^{\boldsymbol{k}}e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i(r+\vartheta)}.\end{aligned} \tag{A.5.64}$$

Therefore, since the parameters $\xi_1,\ldots,\xi_m$ are all different from each other, we conclude that

$$\mathcal{R}_i^1(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \qquad \text{for } \ i = 1,\ldots,m. \tag{A.5.65}$$

STEP 2: $\ell^* \le \boldsymbol{x} \le \boldsymbol{k}$.

Combining similar arguments to the ones used in (A.5.62) with Equation (A.5.59), we derive that

$$\begin{aligned}\sum_{s=1}^{m+1} B_s^+ e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{x}-\ell^*)}\underbrace{\left(\Phi_X\left(\beta_{s,(r+\vartheta)}\right)-(r+\vartheta)\right)}_{=0} + \sum_{u=1}^{n+1} B_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-\boldsymbol{k})}\underbrace{\left(\Phi_X\left(\gamma_{u,(r+\vartheta)}\right)-(r+\vartheta)\right)}_{=0}\\ +\lambda\left(\sum_{i=1}^{m} p_i\xi_i e^{\xi_i(\boldsymbol{x}-k)}\,\mathcal{R}_i^{2,+}(\vartheta;\boldsymbol{k},L,\rho_L) + \sum_{j=1}^{n} q_j\eta_j e^{-\eta_j(\boldsymbol{x}-\ell^*)}\,\mathcal{R}_j^{2,-}(\vartheta;\boldsymbol{k},L,\rho_L)\right) = 0,\end{aligned} \tag{A.5.66}$$

where, for $i = 1,\ldots,m$ and $j = 1,\ldots,n$,

$$\mathcal{R}_i^{2,+}(\vartheta;\boldsymbol{k},L,\rho_L) := -\sum_{s=1}^{m+1} B_s^+\frac{e^{\beta_{s,(r+\vartheta)}(\boldsymbol{k}-\ell^*)}}{\xi_i-\beta_{s,(r+\vartheta)}} - \sum_{u=1}^{n+1}\left(B_u^- - C_u^-\right)\frac{1}{\xi_i-\gamma_{u,(r+\vartheta)}} + \frac{\vartheta e^{\boldsymbol{k}}}{(\xi_i-1)(\delta+\vartheta)} - \frac{\vartheta e^{\boldsymbol{k}}}{\xi_i(r+\vartheta)}, \tag{A.5.67}$$

$$\mathcal{R}_j^{2,-}(\vartheta;\boldsymbol{k},L,\rho_L) := \sum_{s=1}^{m+1}\left(A_s^+\frac{1}{\eta_j+\beta_{s,(r+\vartheta-\rho_L)}} - B_s^+\frac{1}{\eta_j+\beta_{s,(r+\vartheta)}}\right) - \sum_{u=1}^{n+1} B_u^-\frac{e^{-\gamma_{u,(r+\vartheta)}(\boldsymbol{k}-\ell^*)}}{\eta_j+\gamma_{u,(r+\vartheta)}}. \tag{A.5.68}$$

Hence, since the parameters $\xi_1,\ldots,\xi_m,\eta_1,\ldots,\eta_n$ are all different from each other, we conclude that

$$\mathcal{R}_i^{2,+}(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \qquad \text{for } \ i = 1,\ldots,m, \tag{A.5.69}$$

$$\mathcal{R}_j^{2,-}(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \qquad \text{for } \ j = 1,\ldots,n. \tag{A.5.70}$$

STEP 3: $\boldsymbol{k} < \boldsymbol{x} < \infty$.

Following the line of the arguments used in STEP 1 and STEP 2, we rewrite Equation (A.5.60) as

$$\sum_{u=1}^{n+1} C_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-\boldsymbol{k})} \underbrace{\left(\Phi_X\left(\gamma_{u,(r+\vartheta)}\right) - (r+\vartheta)\right)}_{=0}$$
$$+ \underbrace{\left(\frac{\vartheta e^{\boldsymbol{x}}}{\delta+\vartheta} \left(\frac{\sigma^2}{2} + b_X + \lambda\zeta - (r+\vartheta) \right) + (r+\vartheta)\frac{\vartheta e^{\boldsymbol{k}}}{r+\vartheta} - \vartheta e^{\boldsymbol{k}} + \vartheta e^{\boldsymbol{x}} \right)}_{=0} + \lambda \sum_{j=1}^{n} q_j \eta_j e^{-\eta_j(\boldsymbol{x}-\boldsymbol{k})} \mathcal{R}_j^3(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \tag{A.5.71}$$

where, for $j = 1, \ldots, n$,

$$\begin{aligned} \mathcal{R}_j^3(\vartheta;\boldsymbol{k},L,\rho_L) := \sum_{s=1}^{m+1} & \left(A_s^+ \frac{e^{-\eta_j(\boldsymbol{k}-\ell^*)}}{\eta_j + \beta_{s,(r+\vartheta-\rho_L)}} + B_s^+ \frac{e^{\beta_{s,(r+\vartheta)}(\boldsymbol{k}-\ell^*)} - e^{-\eta_j(\boldsymbol{k}-\ell^*)}}{\eta_j + \beta_{s,(r+\vartheta)}} \right) \\ & + \sum_{u=1}^{n+1} \left(B_u^- \frac{1 - e^{-(\eta_j+\gamma_{u,(r+\vartheta)})(\boldsymbol{k}-\ell^*)}}{\eta_j + \gamma_{u,(r+\vartheta)}} - C_u^- \frac{1}{\eta_j + \gamma_{u,(r+\vartheta)}} \right) \\ & - \frac{\vartheta e^{\boldsymbol{k}}}{(\eta_j+1)(\delta+\vartheta)} + \frac{\vartheta e^{\boldsymbol{k}}}{\eta_j(r+\vartheta)}. \end{aligned} \tag{A.5.72}$$

Therefore, since the parameters $\eta_1, \ldots, \eta_n$ are all different from each other, we conclude that

$$\mathcal{R}_j^3(\vartheta;\boldsymbol{k},L,\rho_L) = 0, \qquad \text{for } \; j = 1, \ldots, n. \tag{A.5.73}$$

STEP 4:

To close the system of equations, we impose smooth-fit conditions and obtain the following four identities:

$$\sum_{s=1}^{m+1} \left(A_s^+ - B_s^+\right) - \sum_{u=1}^{n+1} B_u^- e^{-\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)} = 0, \tag{A.5.74}$$

$$\sum_{s=1}^{m+1} B_s^+ e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)} + \sum_{u=1}^{n+1} \left(B_u^- - C_u^-\right) = \frac{\vartheta e^{\boldsymbol{k}}}{\delta+\vartheta} - \frac{\vartheta e^{\boldsymbol{k}}}{r+\vartheta}, \tag{A.5.75}$$

$$\sum_{s=1}^{m+1} \left(A_s^+ \beta_{s,(r+\vartheta-\rho_L)} - B_s^+ \beta_{s,(r+\vartheta)}\right) - \sum_{u=1}^{n+1} B_u^- \gamma_{u,(r+\vartheta)} e^{-\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)} = 0, \tag{A.5.76}$$

$$\sum_{s=1}^{m+1} B_s^+ \beta_{s,(r+\vartheta)} e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)} + \sum_{u=1}^{n+1} \left(B_u^- \gamma_{u,(r+\vartheta)} - C_u^- \gamma_{u,(r+\vartheta)}\right) = \frac{\vartheta e^{\boldsymbol{k}}}{\delta+\vartheta}. \tag{A.5.77}$$

Although we do not further comment on the appropriateness of the smooth-fit conditions (A.5.76) and (A.5.77), we emphasize that smooth-pasting is very natural under hyper-exponential jump-diffusion markets and refer for similar results, e.g. to [CCW10], [XY13], [WZ16].

STEP 5:

To finalize our derivations, we combine the results obtained in STEP 1 - STEP 4. This leads to the following system of equations

$$\mathbf{Q_E}\mathbf{v} = \mathbf{q_E}, \tag{A.5.78}$$

where $\mathbf{v} := (A_1^+, \ldots, A_{m+1}^+, B_1^+, \ldots, B_{m+1}^+, B_1^-, \ldots, B_{n+1}^-, C_1^-, \ldots, C_{n+1}^-)^\intercal$. Here, $\mathbf{q_E} = (\mathbf{q_E^1}, \ldots, \mathbf{q_E^8})^\intercal$ is a $(2m+2n+4)$-dimensional column vector, whose elements are defined in the following way:

i) $\mathbf{q_E^1}$ and $\mathbf{q_E^2}$ are $1 \times m$ vectors given by

$$\left(\mathbf{q_E^1}\right)_i := \frac{\vartheta e^{\boldsymbol{k}} e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i(r+\vartheta)} - \frac{\vartheta e^{\boldsymbol{k}} e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{(\xi_i-1)(\delta+\vartheta)}, \qquad i = 1, \ldots, m, \tag{A.5.79}$$

$$\left(\mathbf{q_E^2}\right)_i := \frac{\vartheta e^k}{\xi_i(r+\vartheta)} - \frac{\vartheta e^k}{(\xi_i-1)(\delta+\vartheta)}, \qquad i = 1, \ldots, m, \tag{A.5.80}$$

ii) $\mathbf{q_E^3}$ and $\mathbf{q_E^4}$ are $1 \times n$ vectors given by

$$\left(\mathbf{q_E^3}\right)_j := 0, \qquad j = 1, \ldots, n, \tag{A.5.81}$$

$$\left(\mathbf{q_E^4}\right)_j := -\frac{\vartheta e^{\boldsymbol{k}}}{\eta_j(r+\vartheta)} + \frac{\vartheta e^{\boldsymbol{k}}}{(\eta_j+1)(\delta+\vartheta)}, \qquad j = 1, \ldots, n, \tag{A.5.82}$$

iii) $\mathbf{q_E^5}$, $\mathbf{q_E^6}$, $\mathbf{q_E^7}$ and $\mathbf{q_E^8}$ are real values given by

$$\mathbf{q_E^5} := 0, \qquad \mathbf{q_E^6} := \frac{\vartheta e^{\boldsymbol{k}}}{\delta+\vartheta} - \frac{\vartheta e^{\boldsymbol{k}}}{r+\vartheta}, \qquad \mathbf{q_E^7} := 0, \qquad \mathbf{q_E^8} := \frac{\vartheta e^{\boldsymbol{k}}}{\delta+\vartheta}. \tag{A.5.83}$$

Finally, $\mathbf{Q_E}$ is a $(2m+2n+4)$-dimensional square matrix

$$\mathbf{Q_E} = \begin{pmatrix} \mathbf{Q_E^{11}} & \mathbf{Q_E^{12}} & \mathbf{Q_E^{13}} & \mathbf{Q_E^{14}} \\ \mathbf{Q_E^{21}} & \mathbf{Q_E^{22}} & \mathbf{Q_E^{23}} & \mathbf{Q_E^{24}} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{Q_E^{81}} & \mathbf{Q_E^{82}} & \mathbf{Q_E^{83}} & \mathbf{Q_E^{84}} \end{pmatrix} \tag{A.5.84}$$

that is defined in the following way:

i) $\mathbf{Q_E^{11}}$, $\mathbf{Q_E^{12}}$ and $\mathbf{Q_E^{13}}$, $\mathbf{Q_E^{14}}$ are respectively $m \times (m+1)$ and $m \times (n+1)$ matrices given, for $i = 1, \ldots, m$, $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{11}}\right)_{is} := -\frac{1}{\xi_i - \beta_{s,(r+\vartheta-\rho_L)}}, \qquad \left(\mathbf{Q_E^{12}}\right)_{is} := \frac{1 - e^{(\beta_{s,(r+\vartheta)}-\xi_i)(\boldsymbol{k}-\ell^*)}}{\xi_i - \beta_{s,(r+\vartheta)}}, \tag{A.5.85}$$

$$\left(\mathbf{Q_E^{13}}\right)_{iu} := \frac{e^{-\gamma_{u,(r+\vartheta)}(\boldsymbol{k}-\ell^*)} - e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i - \gamma_{u,(r+\vartheta)}}, \qquad \left(\mathbf{Q_E^{14}}\right)_{iu} := \frac{e^{-\xi_i(\boldsymbol{k}-\ell^*)}}{\xi_i - \gamma_{u,(r+\vartheta)}}, \tag{A.5.86}$$

ii) $\mathbf{Q_E^{21}}$, $\mathbf{Q_E^{22}}$ and $\mathbf{Q_E^{23}}$, $\mathbf{Q_E^{24}}$ are respectively $m \times (m+1)$ and $m \times (n+1)$ matrices given, for $i = 1, \ldots, m$, $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{21}}\right)_{is} := 0, \qquad \left(\mathbf{Q_E^{22}}\right)_{is} := -\frac{e^{\beta_{s,(r+\vartheta)}(\boldsymbol{k}-\ell^*)}}{\xi_i - \beta_{s,(r+\vartheta)}}, \tag{A.5.87}$$

$$\left(\mathbf{Q_E^{23}}\right)_{iu} := -\frac{1}{\xi_i - \gamma_{u,(r+\vartheta)}}, \qquad \left(\mathbf{Q_E^{24}}\right)_{iu} := -\left(\mathbf{Q_E^{23}}\right)_{iu}, \tag{A.5.88}$$

iii) $\mathbf{Q_E^{31}}$, $\mathbf{Q_E^{32}}$ and $\mathbf{Q_E^{33}}$, $\mathbf{Q_E^{34}}$ are respectively $n \times (m+1)$ and $n \times (n+1)$ matrices given, for $j = 1, \ldots, n$, $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{31}}\right)_{js} := \frac{1}{\eta_j + \beta_{s,(r+\vartheta-\rho_L)}}, \qquad \left(\mathbf{Q_E^{32}}\right)_{js} := -\frac{1}{\eta_j + \beta_{s,(r+\vartheta)}}, \tag{A.5.89}$$

$$\left(\mathbf{Q_E^{33}}\right)_{ju} := -\frac{e^{-\gamma_{u,(r+\vartheta)}(\boldsymbol{k}-\ell^*)}}{\eta_j + \gamma_{u,(r+\vartheta)}}, \qquad \left(\mathbf{Q_E^{34}}\right)_{ju} := 0, \tag{A.5.90}$$

iv) $\mathbf{Q_E^{41}}$, $\mathbf{Q_E^{42}}$ and $\mathbf{Q_E^{43}}$, $\mathbf{Q_E^{44}}$ are respectively $n \times (m+1)$ and $n \times (n+1)$ matrices given, for $j = 1, \ldots, n$, $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{41}}\right)_{js} := \frac{e^{-\eta_j(\boldsymbol{k}-\ell^*)}}{\eta_j + \beta_{s,(r+\vartheta-\rho_L)}}, \qquad \left(\mathbf{Q_E^{42}}\right)_{js} := \frac{e^{\beta_{s,(r+\vartheta)})(\boldsymbol{k}-\ell^*)} - e^{-\eta_j(\boldsymbol{k}-\ell^*)}}{\eta_j + \beta_{s,(r+\vartheta)}}, \tag{A.5.91}$$

$$\left(\mathbf{Q_E^{43}}\right)_{ju} := \frac{1 - e^{-(\eta_j+\gamma_{u,(r+\vartheta)})(\boldsymbol{k}-\ell^*)}}{\eta_j + \gamma_{u,(r+\vartheta)}}, \qquad \left(\mathbf{Q_E^{44}}\right)_{ju} := -\frac{1}{\eta_j + \gamma_{u,(r+\vartheta)}}, \tag{A.5.92}$$

v) $\mathbf{Q_E^{51}}$, $\mathbf{Q_E^{52}}$ and $\mathbf{Q_E^{53}}$, $\mathbf{Q_E^{54}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{51}}\right)_s := 1, \qquad \left(\mathbf{Q_E^{52}}\right)_s := -1, \qquad \left(\mathbf{Q_E^{53}}\right)_u := -e^{-\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)}, \qquad \left(\mathbf{Q_E^{54}}\right)_u := 0, \tag{A.5.93}$$

vi) $\mathbf{Q_E^{61}}$, $\mathbf{Q_E^{62}}$ and $\mathbf{Q_E^{63}}$, $\mathbf{Q_E^{64}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{61}}\right)_s := 0, \qquad \left(\mathbf{Q_E^{62}}\right)_s := e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)}, \qquad \left(\mathbf{Q_E^{63}}\right)_u := 1, \qquad \left(\mathbf{Q_E^{64}}\right)_u := -1, \tag{A.5.94}$$

vii) $\mathbf{Q_E^{71}}$, $\mathbf{Q_E^{72}}$ and $\mathbf{Q_E^{73}}$, $\mathbf{Q_E^{74}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{71}}\right)_s := \beta_{s,(r+\vartheta-\rho_L)}, \qquad \left(\mathbf{Q_E^{72}}\right)_s := -\beta_{s,(r+\vartheta)}, \tag{A.5.95}$$

$$\left(\mathbf{Q_E^{73}}\right)_u := -\gamma_{u,(r+\vartheta)}e^{-\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)}, \qquad \left(\mathbf{Q_E^{74}}\right)_u := 0, \tag{A.5.96}$$

viii) $\mathbf{Q_E^{81}}$, $\mathbf{Q_E^{82}}$ and $\mathbf{Q_E^{83}}$, $\mathbf{Q_E^{84}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \ldots, m+1$, and $u = 1, \ldots, n+1$, by

$$\left(\mathbf{Q_E^{81}}\right)_s := 0, \qquad \left(\mathbf{Q_E^{82}}\right)_s := \beta_{s,(r+\vartheta)}e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{k}-\ell^*)}, \tag{A.5.97}$$

$$\left(\mathbf{Q_E^{83}}\right)_u := \gamma_{u,(r+\vartheta)}, \qquad \left(\mathbf{Q_E^{84}}\right)_u := -\left(\mathbf{Q_E^{83}}\right)_u. \tag{A.5.98}$$

□

Proof of Proposition 5.8. We proceed as in the proof of Proposition 5.7, i.e. we first rewrite the value of the maturity-randomized early exercise premium $\widehat{\mathcal{E}^\star_{\mathcal{DOSC}}}(\cdot)$ as function of the log-price $\boldsymbol{x} := \log(x)$ and of the log-strike $\boldsymbol{k} := \log(K)$ via $\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\cdot)$ by relying on the following relation

$$\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) := \widehat{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, e^{\boldsymbol{x}}; e^{\boldsymbol{k}}, L, \rho_L). \tag{A.5.99}$$

This transforms (5.2.75), (5.2.76) into the following problem

$$\mathcal{A}_X\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - \left(r + \vartheta - \rho_L \mathbb{1}_{(0,L)}(e^{\boldsymbol{k}})\right)\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \qquad \text{for } -\infty < \boldsymbol{x} < b^*, \tag{A.5.100}$$

$$\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = e^{\boldsymbol{x}} - e^{\boldsymbol{k}} - \overline{\mathcal{DOSC}}^\star_E(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L), \qquad \text{for } b^* \le \boldsymbol{x} < \infty, \tag{A.5.101}$$

with $\mathcal{A}_X$ given as in (A.5.57) and b^* denoting the log early exercise boundary, i.e. $b^* := \log(\mathfrak{b}_s)$. Equivalently, this can be written in the following system of three equations

$$\mathcal{A}_X\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - (r + \vartheta - \rho_L)\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \qquad \text{for } -\infty < \boldsymbol{x} < \ell^*, \tag{A.5.102}$$

$$\mathcal{A}_X\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) - (r + \vartheta)\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = 0, \qquad \text{for } \ell^* \le \boldsymbol{x} < b^*, \tag{A.5.103}$$

$$\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = e^{\boldsymbol{x}} - e^{\boldsymbol{k}} - \overline{\mathcal{DOSC}}^\star_E(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L), \qquad \text{for } b^* \le \boldsymbol{x} < \infty, \tag{A.5.104}$$

where we have set $\ell^* := \log(L)$. Consequently, following the arguments in the proof of Proposition 5.7, we obtain that the general solution to (A.5.102)-(A.5.104) takes the following form

$$\overline{\mathcal{E}^\star_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) = \begin{cases} \sum_{s=1}^{m+1} D_s^+ e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(\boldsymbol{x}-\ell^*)}, & -\infty < \boldsymbol{x} < \ell^*, \\ \sum_{s=1}^{m+1} F_s^+ e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{x}-\ell^*)} + \sum_{u=1}^{n+1} F_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-b^*)}, & \ell^* \leq \boldsymbol{x} < b^*, \\ e^{\boldsymbol{x}} - e^{\boldsymbol{k}} - \overline{\mathcal{DOSC}}^\star_E(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L), & b^* \leq \boldsymbol{x} < \infty, \end{cases} \tag{A.5.105}$$

where the coefficients $(D_s^+)_{s=1,\dots,m+1}$, $(F_s^+)_{s=1,\dots,m+1}$, $(F_u^-)_{u=1,\dots,n+1}$ and the free-boundary b^* are subsequently determined by analyzing the solution under the respective equations and in the different regions. Here, following the steps outlined in the proof of Proposition 5.7, we arrive at the following system of equation

$$\mathbf{Q_A}\mathbf{w} = \mathbf{q_A}, \tag{A.5.106}$$

where $\mathbf{w} := (D_1^+, \dots, D_{m+1}^+, F_1^+, \dots, F_{m+1}^+, F_1^-, \dots, F_{n+1}^-)^\intercal$. The vector $\mathbf{q_A} = (\mathbf{q_A^1}, \dots, \mathbf{q_A^6})^\intercal$ is a $(2m+n+3)$-dimensional column vector, whose elements are defined in the following way:

i) $\mathbf{q_A^1}$ and $\mathbf{q_A^2}$ are $1 \times m$ vectors given by

$$\big(\mathbf{q_A^1}\big)_i := \sum_{u=1}^{n+1} C_u^- \frac{e^{-\xi_i(b^*-\ell^*)} e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}}{\xi_i - \gamma_{u,(r+\vartheta)}} + \frac{re^{\boldsymbol{k}} e^{-\xi_i(b^*-\ell^*)}}{\xi_i(r+\vartheta)} - \frac{\delta e^{b^*} e^{-\xi_i(b^*-\ell^*)}}{(\xi_i - 1)(\delta+\vartheta)}, \qquad i = 1, \dots, m, \tag{A.5.107}$$

$$\big(\mathbf{q_A^2}\big)_i := \sum_{u=1}^{n+1} C_u^- \frac{e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}}{\xi_i - \gamma_{u,(r+\vartheta)}} + \frac{re^{\boldsymbol{k}}}{\xi_i(r+\vartheta)} - \frac{\delta e^{b^*}}{(\xi_i - 1)(\delta+\vartheta)}, \qquad i = 1, \dots, m, \tag{A.5.108}$$

ii) $\mathbf{q_A^3}$ is a $1 \times n$ vector given by $\big(\mathbf{q_A^3}\big)_j := 0$, $j = 1, \dots, n$,

iii) $\mathbf{q_A^4}$, $\mathbf{q_A^5}$, $\mathbf{q_A^6}$ are real values given by

$$\mathbf{q_A^4} := 0, \qquad \mathbf{q_A^5} := \frac{\delta e^{b^*}}{\delta+\vartheta} - \frac{re^{\boldsymbol{k}}}{r+\vartheta} - \sum_{u=1}^{n+1} C_u^- e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}, \qquad \mathbf{q_A^6} := 0. \tag{A.5.109}$$

Additionally, $\mathbf{Q_A}$ is a $(2m+n+3)$-dimensional square matrix

$$\mathbf{Q_A} = \begin{pmatrix} \mathbf{Q_A^{11}} & \mathbf{Q_A^{12}} & \mathbf{Q_A^{13}} \\ \mathbf{Q_A^{21}} & \mathbf{Q_A^{22}} & \mathbf{Q_A^{23}} \\ \vdots & \vdots & \vdots \\ \mathbf{Q_A^{61}} & \mathbf{Q_A^{62}} & \mathbf{Q_A^{63}} \end{pmatrix} \tag{A.5.110}$$

that is defined in the following way:

i) $\mathbf{Q_A^{11}}$, $\mathbf{Q_A^{12}}$ and $\mathbf{Q_A^{13}}$ are respectively $m \times (m+1)$ and $m \times (n+1)$ matrices given, for $i = 1, \dots, m$, $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\big(\mathbf{Q_A^{11}}\big)_{is} := -\frac{1}{\xi_i - \beta_{s,(r+\vartheta-\rho_L)}}, \qquad \big(\mathbf{Q_A^{12}}\big)_{is} := \frac{1 - e^{(\beta_{s,(r+\vartheta)} - \xi_i)(b^*-\ell^*)}}{\xi_i - \beta_{s,(r+\vartheta)}}, \tag{A.5.111}$$

$$\big(\mathbf{Q_A^{13}}\big)_{iu} := \frac{e^{-\gamma_{u,(r+\vartheta)}\cdot(b^*-\ell^*)} - e^{-\xi_i(b^*-\ell^*)}}{\xi_i - \gamma_{u,(r+\vartheta)}}, \tag{A.5.112}$$

ii) $\mathbf{Q}^{\mathbf{21}}_{\mathbf{A}}$, $\mathbf{Q}^{\mathbf{22}}_{\mathbf{A}}$ and $\mathbf{Q}^{\mathbf{23}}_{\mathbf{A}}$ are respectively $m \times (m+1)$ and $m \times (n+1)$ matrices given, for $i = 1, \dots, m$, $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\left(\mathbf{Q}^{\mathbf{21}}_{\mathbf{A}}\right)_{is} := 0, \qquad \left(\mathbf{Q}^{\mathbf{22}}_{\mathbf{A}}\right)_{is} := -\frac{e^{\beta_{s,(r+\vartheta)}\cdot(b^*-\ell^*)}}{\xi_i - \beta_{s,(r+\vartheta)}}, \qquad \left(\mathbf{Q}^{\mathbf{23}}_{\mathbf{A}}\right)_{iu} := -\frac{1}{\xi_i - \gamma_{u,(r+\vartheta)}}, \tag{A.5.113}$$

iii) $\mathbf{Q}^{\mathbf{31}}_{\mathbf{A}}$, $\mathbf{Q}^{\mathbf{32}}_{\mathbf{A}}$ and $\mathbf{Q}^{\mathbf{33}}_{\mathbf{A}}$ are respectively $n \times (m+1)$ and $n \times (n+1)$ matrices given, for $j = 1, \dots, n$, $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\left(\mathbf{Q}^{\mathbf{31}}_{\mathbf{A}}\right)_{js} := \frac{1}{\eta_j + \beta_{s,(r+\vartheta-\rho_L)}}, \qquad \left(\mathbf{Q}^{\mathbf{32}}_{\mathbf{A}}\right)_{js} := -\frac{1}{\eta_j + \beta_{s,(r+\vartheta)}}, \tag{A.5.114}$$

$$\left(\mathbf{Q}^{\mathbf{33}}_{\mathbf{A}}\right)_{ju} := -\frac{e^{-\gamma_{u,(r+\vartheta)}\cdot(b^*-\ell^*)}}{\eta_j + \gamma_{u,(r+\vartheta)}}, \tag{A.5.115}$$

iv) $\mathbf{Q}^{\mathbf{41}}_{\mathbf{A}}$, $\mathbf{Q}^{\mathbf{42}}_{\mathbf{A}}$ and $\mathbf{Q}^{\mathbf{43}}_{\mathbf{A}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\left(\mathbf{Q}^{\mathbf{41}}_{\mathbf{A}}\right)_{s} := 1, \qquad \left(\mathbf{Q}^{\mathbf{42}}_{\mathbf{A}}\right)_{s} := -1, \qquad \left(\mathbf{Q}^{\mathbf{43}}_{\mathbf{A}}\right)_{u} := -e^{-\gamma_{u,(r+\vartheta)}\cdot(b^*-\ell^*)}, \tag{A.5.116}$$

v) $\mathbf{Q}^{\mathbf{51}}_{\mathbf{A}}$, $\mathbf{Q}^{\mathbf{52}}_{\mathbf{A}}$ and $\mathbf{Q}^{\mathbf{53}}_{\mathbf{A}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\left(\mathbf{Q}^{\mathbf{51}}_{\mathbf{A}}\right)_{s} := 0, \qquad \left(\mathbf{Q}^{\mathbf{52}}_{\mathbf{A}}\right)_{s} := e^{\beta_{s,(r+\vartheta)}\cdot(b^*-\ell^*)}, \qquad \left(\mathbf{Q}^{\mathbf{53}}_{\mathbf{A}}\right)_{u} := 1, \tag{A.5.117}$$

vi) $\mathbf{Q}^{\mathbf{61}}_{\mathbf{A}}$, $\mathbf{Q}^{\mathbf{62}}_{\mathbf{A}}$ and $\mathbf{Q}^{\mathbf{63}}_{\mathbf{A}}$ are respectively $1 \times (m+1)$ and $1 \times (n+1)$ vectors given, for $s = 1, \dots, m+1$, and $u = 1, \dots, n+1$, by

$$\left(\mathbf{Q}^{\mathbf{61}}_{\mathbf{A}}\right)_{s} := \beta_{s,(r+\vartheta-\rho_L)}, \qquad \left(\mathbf{Q}^{\mathbf{62}}_{\mathbf{A}}\right)_{s} := -\beta_{s,(r+\vartheta)}, \qquad \left(\mathbf{Q}^{\mathbf{63}}_{\mathbf{A}}\right)_{u} := -\gamma_{u,(r+\vartheta)} e^{-\gamma_{u,(r+\vartheta)}\cdot(b^*-\ell^*)}. \tag{A.5.118}$$

Finally, the free-boundary b^* can be recovered by combining (A.5.106) with the usual smooth-fit condition at the boundary level:[25]

$$\sum_{s=1}^{m+1} F_s^+ \beta_{s,(r+\vartheta)} e^{\beta_{s,(r+\vartheta)}\cdot(b^*-\ell^*)} + \sum_{u=1}^{n+1} F_u^- \gamma_{u,(r+\vartheta)} = \frac{\delta e^{b^*}}{\delta+\vartheta} - \sum_{u=1}^{n+1} C_u^- \gamma_{u,(r+\vartheta)} e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}. \tag{A.5.119}$$

□

Proof of Proposition 5.9. To derive Representations (5.3.15) and (5.3.16), we mainly rely on the proof of Proposition 5.8. As earlier, we write

$$\overline{\mathcal{E}^{0,\star}_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) := \widehat{\mathcal{E}^{0,\star}_{\mathcal{DOSC}}}(\vartheta, e^{\boldsymbol{x}}; e^{\boldsymbol{k}}, L, \rho_L), \tag{A.5.120}$$

$$\overline{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DOSC}}}(\vartheta, \boldsymbol{x}; \boldsymbol{k}, L, \rho_L) := \widehat{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DOSC}}}(\vartheta, e^{\boldsymbol{x}}; e^{\boldsymbol{k}}, L, \rho_L), \tag{A.5.121}$$

[25] We emphasize that smooth-fit at the boundary b^* can be proved using the same approach as the one outlined in [Ma20]; See also [PS06], [LM11].

and obtain, by the same arguments as in the proof of Proposition 5.8, that $\overline{\mathcal{E}^{0,\star}_{\mathcal{DOSC}}}(\cdot)$ and $\overline{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DOSC}}}(\cdot)$ take the form

$$\overline{\mathcal{E}^{0,\star}_{\mathcal{DOSC}}}(\vartheta,\boldsymbol{x};\boldsymbol{k},L,\rho_L) = \begin{cases} \sum\limits_{s=1}^{m+1} D_s^{0,+} e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(\boldsymbol{x}-\ell^*)}, & -\infty < \boldsymbol{x} < \ell^*, \\ \sum\limits_{s=1}^{m+1} F_s^{0,+} e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{x}-\ell^*)} + \sum\limits_{u=1}^{n+1} F_u^{0,-} e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-b^*)}, & \ell^* \le \boldsymbol{x} < b^*, \\ e^{\boldsymbol{x}} - e^{\boldsymbol{k}} - \overline{\mathcal{DOSC}}^{\star}_{E}(\vartheta,\boldsymbol{x};\boldsymbol{k},L,\rho_L), & \boldsymbol{x} = b^*, \\ 0, & b^* < \boldsymbol{x} < \infty, \end{cases} \tag{A.5.122}$$

$$\overline{\mathcal{E}^{\mathcal{J},\star}_{\mathcal{DOSC}}}(\vartheta,\boldsymbol{x};\boldsymbol{k},L,\rho_L) = \begin{cases} \sum\limits_{s=1}^{m+1} D_s^{\mathcal{J},+} e^{\beta_{s,(r+\vartheta-\rho_L)}\cdot(\boldsymbol{x}-\ell^*)}, & -\infty < \boldsymbol{x} < \ell^*, \\ \sum\limits_{s=1}^{m+1} F_s^{\mathcal{J},+} e^{\beta_{s,(r+\vartheta)}\cdot(\boldsymbol{x}-\ell^*)} + \sum\limits_{u=1}^{n+1} F_u^{\mathcal{J},-} e^{\gamma_{u,(r+\vartheta)}\cdot(\boldsymbol{x}-b^*)}, & \ell^* \le \boldsymbol{x} < b^*, \\ 0, & \boldsymbol{x} = b^*, \\ e^{\boldsymbol{x}} - e^{\boldsymbol{k}} - \overline{\mathcal{DOSC}}^{\star}_{E}(\vartheta,\boldsymbol{x};\boldsymbol{k},L,\rho_L), & b^* < \boldsymbol{x} < \infty. \end{cases} \tag{A.5.123}$$

Here, analogous derivations to the ones in the proof of Proposition 5.8 show that the vectors of coefficients

$$\mathbf{w_0} := (D_1^{0,+},\dots,D_{m+1}^{0,+},F_1^{0,+},\dots,F_{m+1}^{0,+},F_1^{0,-},\dots,F_{n+1}^{0,-})^{\intercal}$$

and

$$\mathbf{w_J} := (D_1^{\mathcal{J},+},\dots,D_{m+1}^{\mathcal{J},+},F_1^{\mathcal{J},+},\dots,F_{m+1}^{\mathcal{J},+},F_1^{\mathcal{J},-},\dots,F_{n+1}^{\mathcal{J},-})^{\intercal}$$

solve the following system of equations, respectively,

$$\mathbf{Q_A}\mathbf{w_0} = \mathbf{q_{A,0}}, \qquad \text{and} \qquad \mathbf{Q_A}\mathbf{w_J} = \mathbf{q_{A,J}}, \tag{A.5.124}$$

where $\mathbf{q_{A,0}} = (\mathbf{q^1_{A,0}},\dots,\mathbf{q^6_{A,0}})^{\intercal}$ and $\mathbf{q_{A,J}} = (\mathbf{q^1_{A,J}},\dots,\mathbf{q^6_{A,J}})^{\intercal}$ are $(2m+n+3)$-dimensional column vectors, whose elements are defined by:

i) $\mathbf{q^1_{A,0}}$ and $\mathbf{q^2_{A,0}}$ are $1\times m$ vectors given by $\left(\mathbf{q^1_{A,0}}\right)_i := 0,\ \left(\mathbf{q^2_{A,0}}\right)_i := 0,\ i=1,\dots,m,$

ii) $\mathbf{q^3_{A,0}}$ is a $1\times n$ vector given by $\left(\mathbf{q^3_{A,0}}\right)_j := 0,\ j=1,\dots,n,$

iii) $\mathbf{q^4_{A,0}}$, $\mathbf{q^5_{A,0}}$, $\mathbf{q^6_{A,0}}$ are real values given by

$$\mathbf{q^4_{A,0}} := 0, \qquad \mathbf{q^5_{A,0}} := \frac{\delta e^{b^*}}{\delta+\vartheta} - \frac{r e^{\boldsymbol{k}}}{r+\vartheta} - \sum_{u=1}^{n+1} C_u^{-} e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}, \qquad \mathbf{q^6_{A,0}} := 0. \tag{A.5.125}$$

and

i) $\mathbf{q^1_{A,J}}$ and $\mathbf{q^2_{A,J}}$ are $1\times m$ vectors given by

$$\left(\mathbf{q^1_{A,J}}\right)_i := \sum_{u=1}^{n+1} C_u^{-} \frac{e^{-\xi_i(b^*-\ell^*)} e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}}{\xi_i - \gamma_{u,(r+\vartheta)}} + \frac{r e^{\boldsymbol{k}} e^{-\xi_i(b^*-\ell^*)}}{\xi_i(r+\vartheta)} - \frac{\delta e^{b^*} e^{-\xi_i(b^*-\ell^*)}}{(\xi_i-1)(\delta+\vartheta)}, \qquad i=1,\dots,m, \tag{A.5.126}$$

$$\left(\mathbf{q^2_{A,J}}\right)_i := \sum_{u=1}^{n+1} C_u^{-} \frac{e^{\gamma_{u,(r+\vartheta)}\cdot(b^*-\boldsymbol{k})}}{\xi_i - \gamma_{u,(r+\vartheta)}} + \frac{r e^{\boldsymbol{k}}}{\xi_i(r+\vartheta)} - \frac{\delta e^{b^*}}{(\xi_i-1)(\delta+\vartheta)}, \qquad i=1,\dots,m, \tag{A.5.127}$$

ii) $\mathbf{q^3_{A,J}}$ is a $1 \times n$ vector given by $\left(\mathbf{q^3_{A,J}}\right)_j := 0,\ j = 1, \ldots, n$,

iii) $\mathbf{q^4_{A,J}}$, $\mathbf{q^5_{A,J}}$, $\mathbf{q^6_{A,J}}$ are real values given by

$$\mathbf{q^4_{A,J}} := 0, \qquad \mathbf{q^5_{A,J}} := 0, \qquad \mathbf{q^6_{A,J}} := 0. \tag{A.5.128}$$

As a final remark, it is worth mentioning that the above values for $\mathbf{q^5_{A,0}}$ and $\mathbf{q^5_{A,J}}$ only hold under the assumption that $\sigma_X > 0$. In fact, whenever $\sigma_X = 0$, hyper-exponential jump-diffusion processes reduce to finite activity pure jump processes and the corresponding continuous-fit conditions do not anymore hold at the boundary level b^* (cf. [FMV19]). $\square$

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Part III

Curriculum Vitae

Curriculum Vitae

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